ESSENTIAL FAMILIES, MAPPINGS IN DIMENSION THEORY, AND HEREDITARILY INFINITE DIMENSIONAL SPACES

By

EIJI KURIHARA

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By

Eiji Kurihara

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Chairman: James Keesling
Major Department: Mathematics

This paper deals with various properties of separators, continuum-wise separators and essential families. Using these properties, hereditarily infinite dimensional spaces are constructed. The constructions are combinations due to Bing, Rubin, Schori and Walsh. In the first part of the dissertation, it is shown that if $\dim Y < \infty$ and if $f(X) = Y$ is a mapping between compact metric spaces such that $1 \leq m \leq \dim f^{-1}(y) \leq n$ for all $y \in Y$, then there exists a closed set $K \subset X$ such that $\dim K \leq n - m$, and $\dim f(K) = \dim Y$. This answers a question posed by J. Keesling and D. Wilson. The second part of the dissertation deals with hereditarily infinite dimensional spaces. First, a historical survey concerning this problem is given. Next, fundamental properties of separators, continuum-wise separators and essential families are explored. It is shown that the local connectedness of a space is crucial in order that a continuum-wise separator be a separator. It is proved that a pair of disjoint compact sets A and B is essential if and only if there is a quasi-component which meets A and B. It is shown that this explains the totally disconnected Knaster-Kuratowski fan. It is shown that every neighborhood of a continuum-wise separator of a compact Hausdorff space contains a separator. Using this, a continuum-wise separator in a compact metric space is characterized as being a closed set containing a limit of a sequence of separators. Also, a characterization of essential families is derived. That is, a family of pairs of disjoint closed susbsets of a compact metric space is essential if and only if the intersection of any continuum-wise separators is non-empty. In the last part, the important correspondence theorem between essential families and essential maps is mentioned. Using these theorems, alternative proofs of theorems which were previously proved by Henderson, Bing, Rubin, Schori and Walsh are given constructing hereditarily infinite dimensional spaces. It is especially emphasized that the theorems used to prove that every non-degenerate continuum has the dimension > 2 are crucial to construct such spaces. Actual constructions are given in the last part. Finally, the study of the product of essential families is mentioned.

INTRODUCTION

This paper deals with several properties of essential families, separators and continuum-wise separators. Using these properties, we discuss hereditarily infinite dimensional spaces.

In Chapter I, it is shown that if $\dim Y < \infty$ and if $f(X) = Y$ is a mapping between compact metric spaces such that $1 \leq m \leq \dim f^{-1}(y) \leq n$ for all $y \in Y$, then there exists a closed set $K \subset X$ such that $\dim K \leq n - m$ and $\dim f(K) = \dim Y$. This was proved by Kurihara [22] and answered a question posed by J. Keesling and D. Wilson [20].

In Chapter II, we discuss the problem whether there is an infinite-dimensional compact metric space with no n-dimensional compact subsets for any n with $\infty > n \geq 1$. Such a space is said to be hereditarily infinite dimensional. Section 1 of Chapter II is devoted for historical surveys concerning this problem. In Section 2, we derive several properties about essential families, separators, and continuum-wise separators. In Theorem 2.3, we show that local connectedness is necessary in order that a continuum-wise separator be a separator. It is shown in Theorem 2.4 that a pair of disjoint compact sets A and B in a topological space is essential if and only if there is a quasi-component which meets A and B. We indicate that this theorem explains well about the totally disconnected Knaster-Kuratowski fan. In Theorem 2.6, it is proved that any neighborhood of a continuum-wise separator of a compact Hausdorff space contains a separator, from which we derive another characterization of an essential family in a compact metric space using continuum-wise separators. This is Theorem 2.7. In Theorem 2.13, we mention the important correspondence between an essential family and an essential map. Using these theorems, we re-prove the theorems which are proved by Henderson, Bing, Rubin, Schori and Walsh in order to construct hereditarily infinite-dimensional spaces. We show that Theorem 2.11 and Corollary 2.8 are important to prove that every non-degenerate continuum in the hereditarily infinite dimensional compacta constructed by the above authors has dimension ≥ 2. We give actual constructions in Section 3. Finally, in Theorem 2.14 in Section 2, we deal with the problem of the Cartesian product of essential families.

In the Appendix we give a proof of a theorem concerning the limit of a sequence of closed subsets in a compact metric space whose proof allows the shortened proofs here.

CHAPTER I
CLOSED SUBSETS OF THE DOMAIN WHOSE IMAGE HAS THE DIMENSION OF THE RANGE

1. Introduction

Let $f(X) = Y$ be a mapping with X and Y compact metric spaces. In this paper we show that if $1 \le m \le \dim f^{-1}(y) \le n$ for all y in Y, then there is a closed set K in X with $\dim K \le n - m$ and $\dim f(K) = \dim Y$. This is Theorem 1.4 of this paper and our main Theorem. It is related to the general problem of finding relationships between $\dim K$ and $\dim f(k)$ for closed sets K in X. If $\dim X > \dim Y$, Keesling [15] proved that for every $0 \le k \le \dim Y$ there is a closed set K in Y with $\dim K \le k$ and $\dim f^{-1}(K) \ge k + (\dim X - \dim Y)$. He has also investigated whether an analogous Theorem is true for the case of a dimension raising map. He has proved in [18] that if f is open and closed, and $f(X) = Y$ is a dimension raising map of metric spaces, then for every $0 \le k \le \dim X$ there is a closed set K in X with $\dim K \le k$ and with $\dim f(K) \ge k + (\dim Y - \dim X)$. It is natural to ask whether this is true for an arbitrary closed map f. Keesling [16], [17] also deals with the problems of mappings and dimensions.

In 1936 Eilenberg posed the following problem in The Scottish Book [36, Problem #137]. If f maps a compact metric space X onto Y such that $\dim X > \dim Y > 0$, must there be a closed subset K contained in X such that dim $K <$ dim $f(K)$? In reference [21] Kelley gave an affirmative answer to this problem provided that $f(X) = Y$ is open and monotone with nondegenerate inverses and dim $Y < \infty$. However, Keesling and Wilson [19] constructed an n-dimensional continuum X $(n \geq 2)$ in I^{n+1} and a map $f:X \to I = [0,1]$ such that f is onto and if K is any closed set whose dimension is $\leq n - 2$, then $f(K)$ is 0-dimensional. This gave a negative answer to Eilenberg's problem. Rubin, Schori and Walsh [31] also gave a negative solution, but apparently without being aware of the problem in the literature. The main Theorem of this paper (Theorem 1.4) gives a sufficient condition for a positive solution to this problem of Eilenberg.

Theorem 1.4 is an extension of Theorem 2.7 of Keesling and Wilson [20]. The example given by Theorem 5.2 of reference [20] shows that Theorem 1.4 is the best possible. In that example the same hypotheses are satisfied and if K is a closed subset of X with dim $K \leq n - m - 1$, then dim $f(K) = 0$.

The proof of Theorem 1.4 uses techniques found in Kelley [21, Theorem 7.8], Rubin, Schori and Walsh [31, Theorem 6.2], and Keesling and Wilson [20, Theorem 2.7].

We assume that the reader is familiar with standard dimension theory for separable metric spaces. Hurewicz and Wallman [14] is a good reference. All spaces in this chapter are separable metric. We denote the cardinal number of a set A by $|A|$. If Γ denotes a collection of subsets of a set X and $A \subset X$, then $St(A,\Gamma)$ is the collection of all members of Γ which meet A.

The reader needs to be familiar with the notion of essential family [31, Definition 2.2].

2. Properties of Z(P)

Throughout this Section we consider a proper map $f:X \to Y$ between separable metric spaces. Let $P = \{(U_i,V_i)\}_{i=1}^{m}$ be m pairs of open sets of X with $\overline{U}_i \cap \overline{V}_i = \emptyset$ for $i = 1,\dots,m$. Let Z(P) be the set of all points $y \in Y$ such that $f^{-1}(y)$ has an essential family $\{(A_1(y), B_1(y)),\dots,(A_m(y), B_m(y))\}$ such that $A_i(y) \subset \overline{U}_i$ and $B_i(y) \subset \overline{V}_i$.

To study properties of the sets Z(P), we need the following well-known theorems.

Theorem 1.1. In a separable metric space X, any sequence of subsets of X has a convergent subsequence. Here we take the limit of a sequence of subsets of X to be the limit supremum when this is equal to the limit infimum of the sequence. This limit may be empty.

Theorem 1.2. If X is a metric space and M is a subspace of X with $0 \leq \dim M \leq n$, then for every pair A,B of disjoint closed subsets of X there exists a separator S of A and B such that $\dim (S \cap M) \leq n - 1$.

Theorem 1.3. Let M be a subset of a metric space X and A,B a pair of disjoint closed subsets of X. For every separator L' in M between $M \cap A$ and $M \cap B$ there exists a separator L in the space X between A and B such that $M \cap L = L'$.

We also need several lemmas. We omit the proof of the first two lemmas since they are elementary.

Lemma 1.1. If a sequence $\{A_i\}_{i=1}^{\infty}$ of nonvacuous subsets in a metric space X converges to a subset of a compact set A and if for any neighborhood V of A, there is a number n such that $i > n$ implies $A_i \subset V$, then $\lim_{i\to\infty} A_i \neq \emptyset$.

Lemma 1.2. Let f be a closed map of a metric space X into a metric space Y. Let $\{y_i\}$ be a sequence in Y such that $y_i \to y \in Y$. Suppose $A_i \subset f^{-1}(y_i)$, $A_i \neq \emptyset$ and A_i converges to $A \subset f^{-1}(y)$. Then if $f^{-1}(y)$ is compact, then $A \neq \emptyset$.

Lemma 1.3. Let P and $Z(P)$ be as stated above. If $Z(P) \neq 0$, then $Z(P)$ and $f^{-1}(Z(P))$ have the following properties.

(i) $Z(P)$ is a closed set.

(ii) For every $y \in Z(P)$, $f^{-1}(y) \cap (\bigcap_{i=1}^{m} S_i) \neq \emptyset$ for any separators S_i between $\overline{U}_i$ and $\overline{V}_i$ for X. In particular, $\{(\overline{U}_i \cap f^{-1}(Z(P)), \overline{V}_i \cap f^{-1}(Z(P))) \mid i=1,\dots,m\}$ is an essential family in $f^{-1}(Z(P))$.

(iii) Let (B,A) be a pair of closed subsets of $Z(P)$ with $Z(P) \supset B \supset A \neq \emptyset$. Let S_i, $i=1,\dots,m$ be separators between $\overline{U}_i \cap f^{-1}(B)$ and $\overline{V}_i \cap f^{-1}(B)$ in $f^{-1}(B)$. That is, each S_i is a closed subset of $f^{-1}(B)$ such that $f^{-1}(B) - S_i$ is the union of two disjoint open sets U_i' and V_i' in $f^{-1}(B)$ with $\overline{U}_i \cap f^{-1}(B) \subset U_i'$ and $\overline{V}_i \cap f^{-1}(B) \subset V_i'$. Assume there exists a number $\varepsilon > 0$ (depending on S_i's) such that $N_\varepsilon(S_i) = \{x \in X \mid d(x,S_i) < \varepsilon\}$ does not meet $\overline{U}_i \cap f^{-1}(B)$ and $\overline{V}_i \cap f^{-1}(B)$ for all $i = 1,\dots,m$. For any positive numbers α and β with $0 < \alpha < \beta < \varepsilon$, let $C_i = \overline{U_i'} \cap \{x \in f^{-1}(B) \mid d(x,S_i) \geq \beta\}$ and $D_i = \overline{V_i'} \cup \{x \in f^{-1}(B) \mid d(x,S_i) \leq \alpha\}$. Then $f^{-1}(A) \cap C_i$ and $f^{-1}(A) \cap D_i$; $i = 1,\dots,m$ is an essential family for $f^{-1}(A)$.

(iv) Let $\dim f^{-1}(y) \geq m \geq 1$ for all $y \in Y$, where m is as in the definition of P and $Z(P)$. Then there is a set $Z(P)$ such that $\dim Z(P) = \dim Y$. If, in addition, Y is complete, there is a set $Z(P)$ such that $\operatorname{Int} Z(P) \neq \emptyset$.

Proof of (i). This is proved in Keesling and Wilson [20]. For the sake of completeness, we shall reprove this result. Let $\{y_j\}$ be a sequence in $Z(P)$ with $y_j \to y \in Y$. We want to prove that $y \in Z(P)$. For each j, $f^{-1}(y_j)$ has an essential family $\{(A_i(y_j), B_i(y_j))\}_{i=1}^m$ with $A_i(y_j) \subset \overline{U}_i$ and $B_i(y_j) \subset \overline{V}_i$. By Theorem 1.1, there is a subsequence $\{y_{j_k}\}_{k=1}^\infty$ of $\{y_j\}$ such that $A_i(y_{j_k})$ and $B_i(y_{j_k})$ converges. These limits, theoretically, can be empty. However, in the sequel it will follow that they are not. For simplicity, assume $A_i(y_j) \to A_i$ and

$B_i(y_j) \to B_i$ as $j \to \infty$. Then $A_i \subset \overline{U}_i$ and $B_i \subset \overline{V}_i$. We must show that $\{(A_i, B_i)\}_{i=1}^m$ is an essential family. Let S_i; $i = 1, \ldots, m$ be separators of A_i and B_i in $f^{-1}(y)$. Since S_i is a separator of A_i and B_i in $f^{-1}(y)$, we can find a separator S_i' in X such that $X - S_i' = 0_i \cup W_i$; $A_i \subset 0_i$, $B_i \subset W_i$ and $S_i' \cap f^{-1}(y) = S_i$. Since $\lim_{j\to\infty} A_i(y_j) = A_i$ and $\lim_{j\to\infty} B_i(y_j) = B_i$, we claim that

Claim. There is an integer N such that $j > N$ implies $A_i(y_j) \subset 0_i$ and $B_i(y_j) \subset W_i$ for all i.

Proof of Claim. Let us deny the conclusion. Then for some i, $A_i(y_j) \not\subset 0_i$, or $B_i(y_j) \not\subset W_i$ for infinitely many j. For simplicity, assume $A_i(y_j) \not\subset 0_i$ for all j. Then there is a sequence $\{x_j\}$ such that $x_j \in A_i(y_j)$ and $x_j \notin 0_i$. By Theorem 1.1, there is a convergent subsequence $\{\{x_{j_k}\}\}_{k=1}^{\infty}$ of $\{\{x_j\}\}_{j=1}^{\infty}$. Since $f^{-1}(y)$ is compact, the limit is not empty by Lemma 1.2. Since $x_{j_k} \notin 0_i$, the limit is also $\notin 0_i$. This contradicts that $A_i \subset 0_i$. Hence, we have proved our claim.

It follows from our claim that $S_i' \cap f^{-1}(y_j)$ is a separator between $A_i(y_j)$ and $B_i(y_j)$ in $f^{-1}(y_j)$ for $j > N$, which implies that $\bigcap_{i=1}^m S_i' \cap f^{-1}(y_j) \neq \emptyset$.

If necessary, by taking a subsequence, we can assume that $\lim_{j\to\infty}(f^{-1}(y_j) \cap (\bigcap_{i=1}^m S_i'))$ exists. Since $\lim_{j\to\infty} y_j =$

y, $\lim_{j\to\infty}(f^{-1}(y_j)\cap(\bigcap_{i=1}^{m} S_i'))\subset(f^{-1}(y)\cap(\bigcap_{i=1}^{m} S_i'))\subset\bigcap_{i=1}^{m}(f^{-1}(y)\cap S_i') = \bigcap_{i=1}^{m} S_i$. Since $f^{-1}(y)$ is compact, by Lemma 1.2 $\lim_{j\to\infty}(f^{-1}(y_j)\cap(\bigcap_{i=1}^{m} S_i')) \neq \emptyset$. Hence, $\bigcap_{i=1}^{m} S_i \neq \emptyset$. Therefore, $\{(A_i, B_i)\}_{i=1}^{m}$ is an essential family with $A_i \subset \overline{U}_i$, and $B_i \subset \overline{V}_i$. Hence, $y \in Z(P)$ and $Z(P)$ is a closed subset of Y.

Proof of (ii). Since S_i separates $\overline{U}_i$ and $\overline{V}_i$, it also separates $A_i(y)$ and $B_i(y)$ in $f^{-1}(y)$ for any $y \in Z(P)$. Since $\{(A_i(y), B_i(y))\}_{i=1}^{m}$ is an essential family of $f^{-1}(y)$, $\bigcap_{i=1}^{m}(S_i \cap f^{-1}(y)) \neq \emptyset$ for all $y \in Z(P)$. Hence, $(\bigcap_{i=1}^{m} S_i)\cap f^{-1}(y) \neq \emptyset$ for all $y \in Z(P)$.

Proof of (iii). Let S_i; $i = 1,\dots,m$ be separators between $f^{-1}(A)\cap C_i$ and $f^{-1}(A)\cap D_i$ in $f^{-1}(A)$. Since $(f^{-1}(A)\cap C_i)\cup\overline{U}_i$ and $(f^{-1}(A)\cap D_i)\cup\overline{V}_i$ are disjoint closed sets, by Theorem 1.3, S_i can be extended to a separator S_i' between $(f^{-1}(A)\cap C_i)\cup\overline{U}_i$ and $(f^{-1}(A)\cap D_i)\cup\overline{V}_i$ with $S_i'\cap f^{-1}(A) = S_i$. Then $\bigcap_{i=1}^{m} S_i = \bigcap_{i=1}^{m}(S_i'\cap f^{-1}(A)) = \bigcap_{i=1}^{m}(S_i')\cap f^{-1}(A) \neq \emptyset$ by (ii).

Proof of (iv). Let B be a countable basis for X. Consider the family $\wp$ of all $P = \{(U_i, V_i)\}_{i=1}^{m}$ with the property that each U_i and V_i is a finite union of members of B and $\overline{U}_i\cap\overline{V}_i = \emptyset$ for $i=1,\dots,m$. Then the collection $\wp$ is countable. Since f is proper and $\dim f^{-1}(y) > m$ for all $y \in Y$ we have $Y = \bigcup_{P\in\wp} Z(P)$.

By (i), each $Z(P)$ is a closed subset of Y. By the Sum Theorem $\dim Z(P) = \dim Y$ for some P. If Y is complete, then by applying the Baire Category Theorem, $\operatorname{Int} Z(P) \neq \emptyset$ for some P.

This completes the proof of Lemma 1.3.

3. Main Theorem

As stated in the Introduction, Theorem 1.4 is our main Theorem.

Theorem 1.4. If $\dim Y < \infty$ and $f(X) = Y$ is a mapping between compact metric spaces such that $1 \leq m \leq \dim f^{-1}(y) \leq n$ for all $y \in Y$, then there is a closed set $K \subset X$ such that $\dim K \leq n - m$ and $\dim f(K) = \dim Y$. Also, there is a closed set $K_0 \subset X$ such that $\dim K_0 \leq n - m$ and $\operatorname{Int} f(K_0) \neq \emptyset$.

Proof. Let m be a fixed integer satisfying the hypotheses of Theorem 1.4. By Lemma 1.3, we can find a closed set $Z(P)$ of Y such that either $\dim Z(P) = \dim Y$ or $\operatorname{Int} Z(P) \neq \emptyset$ depending on which part of the Theorem we want to prove. Let $P = \{(U_i, V_i)\}_{i=1}^{m}$ and $\overline{U}_i \cap \overline{V}_i = \emptyset$ for $i = 1, \ldots m$. For convenience assume $Y = Z(P)$, $X = f^{-1}(Z(P))$.

Since the dimension of Y is finite, Y can be embedded in E^t for some integer t.

We will prove the Theorem by showing the existence of sequences $\{W_q\}_{q=1}^{\infty}$, $\{J_q\}_{q=1}^{\infty}$ and $\{\mathcal{F}_q\}_{q=1}^{\infty}$ with the following properties:

(1) The sequence $\{W_q\}$ is a nested sequence of open sets of X such that

  (i) $W_q \supset \overline{W}_{q+1}$ for $q = 1,2,3,\ldots,$

  (ii) each open set W_q is covered by a finite collection of open sets $\mathcal{W}_q$, where mesh $\mathcal{W}_q < 1/q$ and the order of $\mathcal{W}_q$ is $\leq n - m + 1$.

(2) Each J_q is a closed subset of X contained in W_q with $f(J_q) = Y$.

(3) $\{\mathcal{F}_q\}_{q=1}^{\infty}$ is a sequence of finite closed coverings of Y which satisfies the following conditions:

  (i) $\mathcal{F}_q = \{F(\alpha_1,\ldots,a_q) \mid \alpha_j \in \Delta_j,\ j = 1,\ldots,q\}$ where $|\Delta_j| < \infty$ for $j = 1,\ldots,q$,

  (ii) $F(\alpha_1,\ldots,\alpha_{q-1}) = \bigcup\{F(\alpha_1,\ldots,\alpha_{q-1},\beta) \mid \beta \in \Delta_q\}$,

  (iii) $|St(F(\alpha_1,\ldots,\alpha_q), \mathcal{F}_q)| < 3^t$ for $q = 1,2,3,\ldots$

(4) (i) Each J_q is a finite union of pairwise disjoint closed sets $\{J_q(\alpha_1,\ldots,\alpha_q) \mid \alpha_j \in \Delta_j,$ $j = 1,\ldots,q\}$ such that $J_q(\alpha_1,\ldots,\alpha_q)$ does not meet $\overline{U}_k$ and $\overline{V}_k$ for all $k = 1,\ldots,m$.

  (ii) Each W_q is a finite union of pairwise disjoint open sets $W_q(\alpha_1,\ldots,\alpha_q)$; $\alpha_j \in \Delta_j$ such that $J_q(\alpha_1,\ldots,\alpha_q) \subset W_q(\alpha_1,\ldots,\alpha_q)$ for $\alpha_j \in \Delta_j$, $j = 1,\ldots,q$.

  (iii) Each $J_q(\alpha_1,\ldots,\alpha_q)$ is an intersection of m separators S_i; $i = 1,\ldots,m$, where S_i separates

$\overline{U}_i \cap f^{-1}(F(\alpha_1,\dots,\alpha_q))$ and $\overline{V}_i \cap f^{-1}(F(\alpha_1,\dots,\alpha_q))$ in $f^{-1}(F(\alpha_1,\dots,\alpha_q))$. Also, $f(J_q(\alpha_1,\dots,\alpha_q)) = F(\alpha_1,\dots,\alpha_q)$.

Under the assumption that properties (1) and (2) are satisfied we can define $K = \bigcap_{q=1}^{\infty} \overline{W}_q$. By properties (1) and (2), $f(K) = Y$. By property (1), $\dim K \leq n - m$.

To begin the induction assume that the diameter of X is less than 1.

Let $J_1 = J_1(1)$ be an intersection $\bigcap_{i=1}^{m} S_i$ of separators S_i for $\overline{U}_i$ and $\overline{V}_i$ in X. Let $\mathcal{F}_1 = F(1) = \{Y\}$, $\Delta_1 = \{1\}$, and $W_1 = X$.

Now it is easy to check that the properties (1)-(4) are satisfied by virtue of Lemma 1.3 (ii).

Assume inductively that there are finite sequences $\{W_i\}_{i=1}^{q}$, $\{J_i\}_{i=1}^{q}$ and $\{\mathcal{F}_i\}_{i=1}^{q}$ which satisfy the properties (1)-(4).

Let $J_q(\alpha_1,\dots,\alpha_q) = \bigcap_{i=1}^{m} S_i(\alpha_1,\dots,\alpha_q)$ be as (4)(iii). Let $f^{-1}(F(\alpha_1,\dots,\alpha_q)) - S_i(\alpha_1,\dots,\alpha_q) = U_i(\alpha_1,\dots,\alpha_q) \cup V_i(\alpha_1,\dots,\alpha_q)$ be a separation with $\overline{U}_i \cap f^{-1}(F(\alpha_1,\dots,\alpha_q)) \subset U_i(\alpha_1,\dots,\alpha_q)$ and $\overline{V}_i \cap f^{-1}(F(\alpha_1,\dots,\alpha_q)) \subset V_i(\alpha_1,\dots,\alpha_q)$ in $f^{-1}(F(\alpha_1,\dots,\alpha_q))$.

By virtue of (4) and the compactness of X, for each $J_q(\alpha_1,\dots,\alpha_q)$ we can find a positive number $\varepsilon = \varepsilon(\alpha_1,\dots,\alpha_q) > 0$ such that $\overline{N}_\varepsilon(J_q(\alpha_1,\dots,\alpha_q)) \subset W_q(\alpha_1,\dots,\alpha_q) \subset W_q$ and $\overline{N}_\varepsilon(S_i(\alpha_1,\dots,\alpha_q))$ does not meet $\overline{U}_i$ and $\overline{V}_i$ for $i = 1,\dots,m$. For $\tau = 1,\dots,3^t$ put

$$I_i(\alpha_1,\ldots,\alpha_q;\tau) = \{x \in U_i(\alpha_1,\ldots,\alpha_q) \mid \varepsilon - (\tau-1)\varepsilon/3^t$$

$$> d(x,S_i(\alpha_1,\ldots,\alpha_q)) > \varepsilon - \tau\varepsilon/3^t\}.$$

Then $I_i(\alpha_1,\ldots,\alpha_q;\tau)$, $\tau = 1,2,\ldots,3^t$ are mutually disjoint open subsets of $f^{-1}(F(\alpha_1,\ldots,\alpha_q))$ such that $\bigcap_{i=1}^m I_i(\alpha_1,\ldots,\alpha_q;\tau) \subset W_q(\alpha_1,\ldots,\alpha_q)$ for $\tau = 1,\ldots,3^t$.

Note that by virtue of Lemma 1.3(iii) for a fixed τ,

$$(C_\tau)_i = \overline{U_i(\alpha_1,\ldots,\alpha_q)} \cap \{x \in f^{-1}(F(\alpha_1,\ldots,\alpha_q)) \mid d(x,S_i(\alpha_1,\ldots,\alpha_q))$$

$$\geq \varepsilon - (\tau-1)\varepsilon/3^t\}$$

and

$$(D_\tau)_i = \overline{V_i(\alpha_1,\ldots,\alpha_q)} \cup \{x \in f^{-1}(F(\alpha_1,\ldots,\alpha_q)) \mid d(x,S_i(\alpha_1,\ldots,\alpha_q))$$

$$\leq \varepsilon - \tau\varepsilon/3^t\},$$

$i = 1,\ldots,m$, is an essential family for $f^{-1}(F(\alpha_1,\ldots,\alpha_q))$ with $\overline{U}_i \cap f^{-1}(F(\alpha_1,\ldots,\alpha_q)) \subset (C_\tau)_i$; and $\overline{V}_i \cap f^{-1}(F(\alpha_1,\ldots,\alpha_q)) \subset (D_\tau)_i$. Also by Lemma 1.3(iii),

(5) $f^{-1}(y) \cap (C_\tau)_i$ and $f^{-1}(y) \cap (D_\tau)_i$; $i = 1,\ldots,m$ is an essential family for $f^{-1}(y)$ for every $y \in F(\alpha_1,\ldots,\alpha_q)$.

Since $\dim f^{-1}(y) \leq n$, by applying Theorem 1.2 m times, for each τ and for each point $y \in F(\alpha_1,\ldots,\alpha_q)$, we can find separators $(S_y)_i(\alpha_1,\ldots,\alpha_q;\tau)$ between $(C_\tau)_i$ and $(D_\tau)_i$ in $f^{-1}(F(\alpha_1,\ldots,\alpha_q))$ with $(S_y)_i(\alpha_1,\ldots,\alpha_q;\tau) \subset I_i(\alpha_1,\ldots,\alpha_q;\tau)$ and $\dim \bigcap_{i=1}^m (S_y)_i(\alpha_1,\ldots,\alpha_q;\tau) \cap f^{-1}(y) \leq n - m$ for

$\tau = 1,\dots,3^t$. Let us notice that $\bigcap_{i=1}^m (S_y)_i(\alpha_1,\dots,\alpha_q;\tau) \subset \bigcap_{i=1}^m I_i(\alpha_1,\dots,\alpha_q;\tau) \subset W_q(\alpha_1,\dots,\alpha_q)$. Let $J_y(\alpha_1,\dots,\alpha_q;\tau) = \bigcap_{i=1}^m (S_y)_i(\alpha_1,\dots,\alpha_q;\tau)$. Then

(6) $\dim J_y(\alpha_1,\dots,\alpha_q;\tau) \cap f^{-1}(y) < n - m$, $J_y(\alpha_1,\dots,\alpha_q;\tau) \subset I_i(\alpha_1,\dots,\alpha_q;\tau)$ and $J_y(\alpha_1,\dots,\alpha_q;\tau) \subset W_q(\alpha_1,\dots,\alpha_q)$.

Note that by (5),

(7) for each $z \in F(\alpha_1,\dots,\alpha_q)$ $f^{-1}(z) \cap J_y(\alpha_1,\dots,\alpha_q;\tau) \neq \emptyset$.

It follows from (6) that for every τ and $y \in F(\alpha_1,\dots,\alpha_q)$ we can choose an open set $W_y(\alpha_1,\dots,\alpha_q;\tau)$ of X such that

(8) $J_y(\alpha_1,\dots,\alpha_q;\tau) \cap f^{-1}(y) \subset W_y(\alpha_1,\dots,\alpha_q;\tau)$ $\overline{W_y(\alpha_1,\dots,\alpha_q;\tau)} \cap I_i(\alpha_1,\dots,\alpha_q;\tau') = \emptyset$ for $\tau \neq \tau'$ and for all i, and $\overline{W_y(\alpha_1,\dots,\alpha_q;\tau)} \subset W_q(\alpha_1,\dots,\alpha_q)$,

and moreover we can require that it is covered by a finite collection of open sets $\mathcal{W}_y(\alpha_1,\dots,\alpha_q;\tau)$ with $W_y(\alpha_1,\dots,\alpha_q;\tau) = \bigcup \mathcal{W}_y(\alpha_1,\dots,\alpha_q;\tau)$, mesh $\mathcal{W}_y(\alpha_1,\dots,\alpha_q;\tau) < 1/(q+1)$ and the order of $\mathcal{W}_y(\alpha_1,\dots,\alpha_q;\tau) < n - m + 1$. By virtue of (8), if we let $\mathcal{W}_y(\alpha_1,\dots,\alpha_q) = \bigcup_{\tau=1}^{3^t} \{\mathcal{W}_y(\alpha_1,\dots,\alpha_q;\tau)\}$ then we have ord $\mathcal{W}_y(\alpha_1,\dots,\alpha_q) < n - m + 1$. Let $W_y(\alpha_1,\dots,\alpha_q) = \bigcup \mathcal{W}_y(\alpha_1,\dots,\alpha_q)$. Let $J_y(\alpha_1,\dots,\alpha_q) = \bigcup_{\tau=1}^{3^t} J_y(\alpha_1,\dots,\alpha_q;\tau) \cap f^{-1}(F(\alpha_1,\dots,\alpha_q))$ and $g_y = f|J_y(\alpha_1,\dots,\alpha_q)$.

Since g_y is a closed map, there is an open set $V_y \ni y$ of Y such that $g_y^{-1}(V_y) \subset W_y(\alpha_1,\dots,\alpha_q)$. Since Y (hence, also $F(\alpha_1,\ldots,\alpha_q))$ is a compact subset of E^t, there is a finite closed cover $\mathcal{F}_{q+1} = \{F(\alpha_1,\ldots,\alpha_{q+1}) | \alpha_j \in \Delta_j,\ j = 1, \ldots, q + 1\}$ and $\Gamma = \{F(\alpha_1,\ldots,\alpha_q,\alpha_{q+1}) | \alpha_{q+1} \in \Delta_{q+1}\}$ of Y and $F(\alpha_1,\ldots,\alpha_q)$ respectively with the property that Γ refines the collection $\{V_y | y \in F(\alpha_1,\ldots,\alpha_q)\}$ for each $(\alpha_1,\ldots,\alpha_q)$, $F(\alpha_1,\ldots,\alpha_q) = \bigcup_{\alpha_{q+1}} F(\alpha_1,\ldots,\alpha_{q+1})$ and $|St(F(\alpha_1,\ldots,\alpha_{q+1}), \mathcal{F}_{q+1})| < 3^t$. Let $Y_1 = F(\alpha_1,\ldots,\alpha_q,1),\ldots,Y_s = F(\alpha_1,\ldots,\alpha_q,s)$ be a listing of all the members of Γ, and $Y_i \subset V_{y_i}$ for $i = 1,\ldots,s$.

Let $K_1 = J_{y_1}(\alpha_1,\ldots,\alpha_q;1) \cap g_{y_1}^{-1}(Y_1)$. Assume inductively that there are pairwise disjoint closed sets $K_1,\ldots,K_r$

$$K_i = J_{y_i}(\alpha_1,\ldots,\alpha_q;\tau) \cap g_{y_i}^{-1}(Y_i)$$

for some choice of $\tau \in \{1,2,\ldots,3^t\}$. Since $|St(Y_{r+1},\Gamma)| < 3^t$ and since there are 3^t sets $J_{y_{r+1}}(\alpha_1,\ldots,\alpha_q;1),\ldots,J_{y_{r+1}}(\alpha_1,\ldots,\alpha_q;3^t)$, we can find an integer t' with the property that if $Y_i \in St(Y_{r+1},\Gamma)$ and $i < r + 1$, then $J_{y_{r+1}}(\alpha_1,\ldots,\alpha_q;t') \cap g_{y_{r+1}}^{-1}(Y_{r+1})$ misses K_i. Let $K_{r+1} = J_{y_{r+1}}(\alpha_1,\ldots,\alpha_q;t') \cap g_{y_{r+1}}^{-1}(Y_{r+1})$.

If we let $J_{q+1}(\alpha_1,\ldots,\alpha_q,\alpha_{q+1}) = K_{\alpha_{q+1}}$, $\alpha_{q+1} \in \Delta_{q+1} = \{1,\ldots,s\}$, then each $J_{q+1}(\alpha_1,\ldots,\alpha_q,\alpha_{q+1})$ is an intersection of m-separators between $\bar{U}_i \cap f^{-1}(F(\alpha_1,\ldots,\alpha_q,\alpha_{q+1}))$ and $\bar{V}_i \cap f^{-1}(F(\alpha_1,\ldots,\alpha_{q+1}))$; $i = 1,\ldots,m$, and $f(J_{q+1}$

$(\alpha_1,\dots,\alpha_q,\alpha_{q+1})) = F(\alpha_1,\dots,\alpha_{q+1})$ by (7). By the construction above, $J_{q+1}(\alpha_1,\dots,\alpha_q,\alpha_{q+1})$, $\alpha_j \in \Delta_j$, $j = 1,\dots,q+1$ are pairwise disjoint closed sets such that $J_{q+1}(\alpha_1,\dots,\alpha_{q+1})$ does not meet $\overline{U}_k$ and $\overline{V}_k$ for all $k = 1,\dots,m$.

The open set W_{q+1} can be found in the following way. Each $J_{q+1}(\alpha_1,\dots,\alpha_q,\alpha_{q+1})$ is covered by one of the open sets $W_{y_i}(\alpha_1,\dots,\alpha_q)$ which is also covered by a family of open sets whose order is $< n - m + 1$ and mesh $< 1/(q+1)$.

Since the members of $\{J_{q+1}(\alpha_1,\dots,\alpha_{q+1}) \mid \alpha_j \in \Delta_j, j = 1,\dots,q+1\}$ are a finite number of pairwise disjoint closed sets, we can find the same number of pairwise disjoint open sets each of which contains exactly one of the previous members $J_{q+1}(\alpha_1,\dots,\alpha_{q+1})$.

By taking the intersection of all pairs of open sets where the first comes from the first family and the second from the second, we obtain a family $\mathcal{W}_{q+1} = \bigcup\{\mathcal{W}_{q+1}(\alpha_1,\dots,\alpha_{q+1}) \mid \alpha_j \in \Delta_j, j = 1,\dots,q+1\}$ such that $W_{q+1}(\alpha_1,\dots,\alpha_{q+1}) = \bigcup \mathcal{W}_{q+1}(\alpha_1,\dots,\alpha_{q+1})$ satisfies the condition (4)(ii) and ord $\mathcal{W}_{q+1} < n - m + 1$ and mesh $\mathcal{W}_{q+1} < 1/(q+1)$. Since each $\overline{W_y(\alpha_1,\dots,\alpha_q)}$ is contained in $W_q(\alpha_1,\dots,\alpha_q) \subset W_q$, we have $W_q \supset \overline{W}_{q+1}$.

Now,

$$f(J_{q+1}) = f\Big(\bigcup_{a_1,\dots,a_{q+1}} J_{q+1}(\alpha_1,\dots,\alpha_{q+1})\Big) = \bigcup_{\alpha_1,\dots,\alpha_{q+1}}$$

$$f(J_{q+1}(\alpha_1,\dots,\alpha_{q+1})) = \bigcup_{\alpha_1,\dots,\alpha_{q+1}} F(\alpha_1,\dots,\alpha_{q+1}) = Y.$$

Thus, we have completed the proof of the Theorem.

In Theorem 1.4 we found a set K with $\dim f(K) = \dim Y$ and $\dim K \leq n - m$. The following example shows that we cannot find a closed set K with $\dim K \leq n - m$ and $f(K) = Y$.

Example. Let a positive number ε be given. In E^2, consider $X = I \cup A$, where $I = \{(0,y) \in E^2 \mid -1 \leq y \leq 1\}$ and $A = \{(x,y) \in E^2 \mid \sin 1/x - x\varepsilon \leq y \leq \sin 1/x + x\varepsilon,\ 0 < x \leq 1\}$. Consider $Y = \{x \in E \mid 0 \leq x \leq 1\}$ and a projection $f: X \to Y$ defined by $f(x,y) = x$. It is easy to check that $\dim f^{-1}(y) = 1$ for all $y \in Y$. By Theorem 1.4 there is a closed set K with $\dim K = 0$ and $\dim f(K) = \dim Y = 1$. On the other hand, there is no closed 0-dimensional set K_0 with $f(K_0) = Y$. Because any closed set K_0 with $f(K_0) = Y$ must contain I and therefore must be at least one-dimensional.

CHAPTER II
FINITE DIMENSIONAL SUBSETS OF INFINITE DIMENSIONAL SPACES

1. Historical Survey

A metric space X is said to be strongly infinite dimensional provided it possesses a countable essential family $\{(A_k, B_k): k = 1,2,\ldots\}$. A space X is weakly infinite-dimensional if every countable sequence of pairs $\{(A_k, B_k): k = 1,2,\ldots\}$ of disjoint closed sets in X is not an essential family. If a space X is the union of a countable number of 0-dimensional subsets, then X is said to be countable-dimensional. We say that a space is hereditarily infinite-dimensional if it is infinite-dimensional and contains no n-dimensional closed subsets for each $1 \leq n < \infty$.

In 1926, L.A. Tumarkin [35] proposed the problem whether every infinite-dimensional metrizable compactum contains subcompacta of every finite-dimension, but Mazurkiewicz, in 1933, first posed the question in print [24]. In 1946, A. van Heemert claimed [6] to have shown that for a compact metric space of infinite dimension there must exist 1- and 2-dimensional compact subsets. But D.W. Henderson showed that arguments used in this paper are incorrect in reference [10]. Henderson [10] provides a good historical survey concerning this topic up to that date. Related to our topics, Mazurkiewicz [25] constructed n-dimensional spaces such that all comapct subsets are 0-dimensional. These are the totally disconnected n-dimensional spaces and the compact subspaces are zero-dimensional because any totally disconnected compact space has dimension zero ([14], page 20). In 1932, Hurewicz [13] showed, assuming the continuum hypothesis, that there is a space in the Hilbert Cube each finite-dimensional subset of which is countable. This is the first example of a hereditarily infinite-dimensional space. Also, under the continuum hypothesis, Tumarkin ([34], 1961; [35], 1963) showed that every strongly infinite-dimensional space contains hereditarily strongly infinite-dimensional subsets whose subsets are either countable or strongly infininte dimensional.

Without the continuum hypothesis, D.W. Henderson [7], in 1965, announced the first construction of a hereditarily strongly infinte-dimensional compactum. An improved version was presented by him in [8]. R.H. Bing [3] gave a further simplification. Henderson [9] gave a modification of the construction in [8] to prove that every strongly infinite-dimensional compactum contains a hereditarily strongly infinite-dimensional compact subset. This was also done without the continuum hypothesis. Zarelua ([41], 1972; [40], 1974) gave constructions of hereditarily strongly infinite-dimensional compacta using a different approach. However, these examples are still difficult for the third person to understand. Hence, thereafter, several mathematicians tried to improve the construction of a hereditarily strongly infinite-dimensional compactum.

In 1977, Rubin, Schori and Walsh [31] developed simpler axiomatic techniques to this type construction. Using these techniques, Rubin ([28], 1980; [29], 1980), Walsh ([39], 1979), and Schori and Walsh ([33], 1978) gave simpler constructions of hereditarily strongly infinite-dimensional spaces. The hereditarily strongly infinite-dimensional compacta in Walsh [39] and Rubin [28] have the additional property that the dimension of each subset (not necessarily closed subset) is either 0 or infinity. Walsh [39] also gave an outline of proof that every strongly infinite-dimensional compactum contains an infinite dimensional subcompactum which contains no n-dimensional subsets for $\infty > n \geq 1$. Rubin [29] showed that any strongly infinite-dimensional separable metric space contains a hereditarily strongly infinite-dimensional subspace all of whose subsets are either 0-dimensional or strongly infinite-dimensional. So far, Schori and Walsh [33] have presented the most readable constructions and proofs of hereditarily strongly infinite-dimensional compacta, and the most reasonable proofs that every strongly infinite-dimensional compact metric space contains a hereditarily strongly infinite-dimensional subspace.

Every countable dimensional space X is weakly infinite dimensional. Since if $X = \bigcup_{i=1}^{\infty} X_i$, with $\dim X_i \leq 0$, $i = 1,2, \ldots$, and if $\{(A_i,B_i)\}_{i=1}^{\infty}$ is a countable sequence of pairs of disjoint closed sets in X, then there are separators S_i

between A_i and B_i disjoint from X_i for all i. Hence,

$$\bigcap_{j=1}^{\infty} S_j = (\bigcap_{j=1}^{\infty} S_j)\cap X = (\bigcap_{j=1}^{\infty} S_j)\cap(\bigcup_{i=1}^{\infty} X_i) =$$

$$= \bigcup_{i=1}^{\infty}((\bigcap_{j=1}^{\infty} S_j)\cap X_i) \subset \bigcup_{i=1}^{\infty}(S_i\cap X_i) = \emptyset.$$

By contraposition, every strongly infinite-dimensional space X is not countable dimensional. Aleksandrov is the first to have posed the question whether the converse is true for compacta, i.e., whether each non-countable dimensional compactum is strongly infinite-dimensional. This was stated in Aleksandrov's preface to the Russian 1948 translation of Hurewicz and Wallman's book Dimension Theory [14]. This question was repeated by him ([2], 1951, §4, Hypothesis 1; [1], 1960) and by others (Henderson [11], 1967, Question 2; Nagami [26], 1970, Problem 13-7). This old and famous problem of Aleksandrov was finally solved negatively by Roman Pol ([27], 1981). Since this construction is so simple, we shall describe it briefly. He noticed that a construction in reference [31] assures the existence of a topologically complete space X which is totally disconnected and strongly infinite-dimensional. Rubin [30] gave a good description of such a space X. Then there exists a compactification cX of X such that the remainder cX\X is countable dimensional ([23]; as pointed out by American Mathematical Review (32: 4651) of Katětov, there are errors in this paper, but these errors do not affect the proof of this fact). This cX is a weakly infinite-dimensional but not countable-dimensional compactum. To check that cX is weakly infinite-dimensional, let $\{(A_i,B_i)\colon i = 0,1,\ldots\}$ be a countable sequence of pairs of disjoint closed sets in cX, and let $cX\setminus X = \bigcup_{i=1}^{\infty} X_i$ with $\dim X_i \leqq 0$. There are separators S_i between A_i and B_i disjoint from X_i, $i = 1,2,\ldots$. Then $S = \bigcap_{i=1}^{\infty} S_i$ is a compact subset of the totally disconnected space X; hence, $\dim S \leqq 0$. Therefore, there is a separator S_0 in cX between A_0 and B_0 disjoint from S. Thus, $\bigcap_{i=0}^{\infty} S_i = \emptyset$; hence, cX is a weakly infinite-dimensional compactum. If cX were countable dimensional, X would be too. But X is not since every strongly infinite-dimensional space is not countable dimensional. This proves that cX is a non-countable-dimensional, weakly infinite-dimensional compactum.

Let $K_n = K(Z,n)$ be a CW-complex, called an Eilenberg-MacLane classification space, obtained from the n-sphere S^n by attaching cells having dimensions $> n + 2$ in such a way that the m-th homotopy group $\pi_m(K_n)$ is zero for all $m > n + 1$. Let $H^m(X,A\colon Z)$ be an m-th Čech cohomology of a pair (X,A) with A a closed subset of X. A metric space X has a cohomological dimension $< n$ with respect to integer coefficients Z, written c-dim $X < n$ if the space X satisfies one of the following equivalent conditions:

(i) For any closed subset A of X, $H^m(X,A\colon Z) = 0$ for every $m > n + 1$,

(ii) For any closed subset A of X and the inclusion $i:A \subset X$, the induced homomorphism $i^*:H^m(X:Z) \to H^m(A;Z)$ is onto for any $m \geq n$,

(iii) Each map $f:A \to K_n$, from a closed subset into an Eilenberg-Maclane space $K_n = K(Z,n)$, extends to a map $\overline{f}:X \to K_n$.

Then c-dim $X = n$ if c-dim $X \leq n$ and c-dim $X \leq n - 1$ is false.

The cohomological dimensions defined by (i) and (ii) on a general topological space X are called, respectively, the large cohomological dimension and the small cohomological dimension with integer coefficients Z. If the space X is paracompact Hausdorff, then the defintions (i) and (ii) coincide ([26], Theorem 37-7). Hence, since we consider a metric space X, we have no problem no matter which definition we use.

Concerning the relationship between the dimension and the cohomological dimension of a metric space X, we have c-dim $X \leq$ dim X and c-dim X = dim X if a space X has a finite dimension. Hence, it is natural to ask whether there exists an infinite dimensional compact metric space with finite cohomological dimension.

A cell-like map $f:X \to Y$ between metric spaces is a proper map such that $f^{-1}(y)$ has the shape of a point for each $y \in Y$. A metric space A has a shape of point or trivial shape if every map of A to a polyhedron (or a metric ANR) is null homotopic.

An important theorem related to the relationship with dimension, cohomological dimension, and cell-like maps is announced by R.D. Edwards [[4], 1978) and its proof is found in a paper of J.J. Walsh ([38], Section 6), which states that a compact metric space X has cohomological dimension $\leq n$ if and only if X is a cell-like image of some compact metric space Y with $\dim Y \leq n$. Hence, if a compactum X with $\dim X = \infty$ and $\text{c-dim } X < \infty$ existed, then such an X would be the image of a cell-like dimension raising map.

There is a major unsolved problem which has motivated some work in constructing examples of infinite-dimensional compact metric spaces that contain no n-dimensional closed subsets for $\infty > n > 1$. This problem is whether or not there is a cell-like dimension raising mapping. If $f:X \to\to Y$ with $\dim X < \infty$ is a cell-like dimension raising mapping between compacta, then by the theorem of R.D. Edwards mentioned above and by the fact that $\dim X = \text{c-dim } X$ for a finite dimensional space, we have $\dim Y = \infty$ and Y has finite cohomological dimension. If Y has a closed subset of each finite dimension then the cohomological dimension of Y must be infinity; hence, the space Y must have the property that any closed subset has either a dimension $< n$ for some fixed integer n or infinity. Hence, hereditarily strongly infinite-dimensional compacta are such candidates.

We finish this survey by noticing that essential families played an important role in the constructions of Rubin, Schori and Walsh. It seems that essential families may have more applications in the future.

2. Separators, Continuum-wise Separators and Essential Families

In this section we will explore various properties and relationships of separators, continuum-wise separators and essential families. We will include almost all works concerning these properties done by Rubin, Schori and Walsh and try to discuss further details. We prove theorems from a slightly different point of view. Hence, we intend this section to be a summary of these properties and to clarify relationships between them.

A non-empty set $K \subset X$ is called a quasi-component of a topological space X if K can be expressed as the intersection of open-and-closed sets and if for every open-and-closed set $U \subset X$ with $K \cap U \neq \emptyset$ we have $K \subset U$. A quasi-component always contain the components. When we consider a separator of a compact metric space, the following theorem is fundamental ([5], Lemma 1.4.4.).

Theorem 2.1. In every compact Hausdorff space quasi-components and components coincide.

We mean by a continuum a non-empty compact connected Hausdorff space.

Definition 2.1. A closed subset S of a topological space X is said to continuum-wise separate A and B in X provided every continuum in X from A to B meets S.

The following facts follow from the definition. If S is a continuum-wise separator of A and B, then any closed set S' which contains S is a continuum-wise separator. A or B itself is a continuum-wise separator. Moreover, if either A of B is empty, then any subset of X is a continuum-wise separator, because in this case, the hypothesis of the definition of the continuum-wise separator is automatically satisfied since no continuum meets both A and B. This contrasts with the fact that $\{A,B\}$ is not an essential family if one of them is empty since, in this case, the empty set separates them.

We have the following relationship between a separator and a continuum-wise separator.

Lemma 2.1. A separator S of disjoint closed sets A and B in X is a continuum-wise separator.

Proof. If one of A and B is empty, then the lemma is trivially true. Hence, we suppose that A and B are not empty. Assume C is a continuum which meets A and B but fails to meet S. Since S is a separator, by definition, there are non-empty open sets U_1 and U_2 in X such that $X \setminus S = U_1 \cup U_2$ with $A \subset U_1$ and $B \subset U_2$. It follows from $C \subset X \setminus S$ that $C \cap U_1$ and $C \cap U_2$ give a separation of C by the empty set. This contradicts the connectedness of a continuum C. Hence, every continuum which meets A and B must meet B; i.e., S is a continuum-wise separator.

We are interested in the converse of this lemma. Suppose S is a continuum-wise separator between A and B which misses both A and B. Does it follow that S is a separator? In the sequel, we give negative answers in several situations. We give the first negative case in the following.

Example 2.1. Let X' be the Knaster-Kuratowski fan constructed in an Euclidean plane E^2 as follows. Let C be the usual Cantor ternary set on the unit interval [0,1] lying on the x axis. Let a be the point in the plane whose coordinates are ($\frac{1}{2}$, 1). Let P be the set of all end points in C and Q = C - P. For any point $c \in C$, let L(c) be the line segment joining c and a. Define L'(c) as follows:

$$L'(c) = \begin{cases} \{(x,y) \in L(c) \mid y \text{ is rational}\} & \text{if } c \in P, \\ \{(x,y) \in L(c) \mid y \text{ is irrational}\} & \text{if } c \in Q. \end{cases}$$

Set $X' = \bigcup\{L'(c) \mid c \in C = P \cup Q\}$ and $X = X' - \{a\}$. Then we know that X is a totally disconnected separable metric space under the induced metric from the Euclidean space E^2 with dim X = 1. We also know that X' is a 1-dimensional connected separable metric space [14]. Given three numbers $0 < n_1 < n_2 < n_3 < 1$. Then $A = \{(x,y) \in X \mid y = n_1\}$ and $B = \{(x,y) \in X \mid y = n_3\}$ are disjoint closed sets of X and $S = \{(x,y) \in X \mid y = n_2\}$ is a separator of A and B in X. However, any closed subset S' of X is a continuum-wise separator since there is no continuum connecting A and B. Therefore, the hypothesis of the definition of a continuum-wise separator is automatically satisfied. It follows that we can find a subset S' of X which does not meet A and B such that S' is a continuum-wise separator but not a separator. For this sake, it is enough to take S' as the empty set. Then the method of proof that X' is connected can apply here to insure that the pair $\{A,B\}$ is an essential family; hence, no empty set separates A and B in X.

The following theorem is very convenient when we consider the essential family in a compact Hausdorff space.

Theorem 2.2. Let X be a compact Hausdorff space, then a pair of nonempty disjoint closed sets A and B is an essential family if and only if there is a continuum C connecting A and B.

Proof. Sufficiency: Assume there is a continuum C connecting A and B. Let S be a separator between A and B. If $S \cap C = \emptyset$, then it gives a separation of $A \cap C$ and $B \cap C$ in C by the empty set, which contradicts the connectedness of C. Hence, $S \supset S \cap C \neq \emptyset$, i.e., $\{A,B\}$ is an essential family.
Necessity: Conversely assume there is no continuum connecting A and B. Let $C(A) = \{A_i \mid i \in \Delta\}$ be sets of all components (maximal connected sets) of X which meet A. Then none of the components meets B by our hypothesis since every component in a compact set is a continuum. Pick an arbitrary element $a \in A$. Then there is a component $A_i \in C(A)$ with $a \in A_i$. Since every component A_i is a quasi-component by Theorem 2.1, it is an intersection of open-and-closed sets in X. Since

A_i is contained in an open set $X\setminus B$, by the compactness of X, there is an open-and-closed set 0_i with $a \in A_i \subset 0_i \subset X\setminus B$. Hence, A is covered by such open-and-closed sets $\{0_i\}$. By the compactness of A, A is covered by a finite number of such sets, say $\{0_j\}_{j=1}^n$. Then $0 = \bigcup_{j=1}^{n} 0_j$ is an open-and-closed set with $A \subset 0 \subset X\setminus B$. Hence, $\{0, X\setminus 0\}$ gives a separation of $\{A,B\}$ by the empty set. Hence, $\{A,B\}$ is not an essential family. This proves the necessity.

Remark 2.1. In the proof of Theorem 2.2, we use only the fact that some connected subsets of X meet both A and B. Hence, it is true that the disjoint closed pair $\{A,B\}$ in a compact Hausdorff space X is an essential family if and only if A and B are connected by some connected subset of X. In this case, by considering the maximal connected subset of X which meets A and B, we can always find a continuum connecting both A and B. However, A and B in the space X of Example 2.1 exhibits what happens outside the compact metric spaces. In this example there is no continuum nor no connected subset meeting both A and B. Notice that A and B are not compact. Unless the numbers n_1 and n_3 are simultaneously either rational or irrational, there is no quasi-component meeting both A and B because only quasi-components in X are the forms: $L'(c) - \{a\}$, $c \in C$, which intersect both A and B if and only if n_1 and n_3 are simultaneously rational or irrational. Two different points p and q lying on the same broken segment $L'(c) - \{a\}$ form an essential family since the method of proof that X' is connected can again apply in this case. Notice that $\{p\}$ and $\{q\}$ are compact. Our Theorem 2.4 stated later assures the existence of a quasi-component meeting both $\{p\}$ and $\{q\}$.

Corollary 2.1. Let X be a continuum. Then every pair of non-empty disjoint closed subsets A and B is an essential family.

Proof. A and B are connected by a continuum X. Hence, we have the corollary by Theorem 2.2.

Corollary 2.2. Let X be a continuum, A a non-empty closed subset and U an open neighborhood of A in X such that $\partial U = \overline{U}\setminus U \neq \emptyset$. Then a pair $\{A, \partial U\}$ is an essential family in $\overline{U}$; hence, there is a continuum connecting A and ∂U in $\overline{U}$.

Proof. Assume not. Then there exists a subset $V \subset \overline{U}$ which is open-and-closed in $\overline{U}$ such that $A \subset V$ and $V \cap \partial U = \emptyset$. V is closed in a closed set $\overline{U}$, hence, V is closed in X. Since $V \subset \overline{U}\setminus\partial U = U$, V is open in an open set U; hence, V is open-and-closed in X. Therefore, $\{V, X\setminus V\}$ gives a separation of A and ∂U in X by the empty set. This contradicts that $\{A, \partial U\}$ is an essential family in X.

Let S be a continuum-wise separator of non-empty closed sets A and B in a continuum X. Assume S misses A and B. Is it true that S is a separator? Again, we give a negative answer in the following example.

Example 2.2. Consider a continuum X defined in an Euclidean plane E^2 defined in the following way:

Let $I = \{(x,y) \in E^2 | x = 0, -1 \leq y \leq 1\}$,

$Y = \{(x,y) \in E^2 | y = \sin \frac{1}{x}, 0 < x \leq \frac{1}{\pi}\}$

and $X = I \cup Y$. (See Figure 1.)

Let $A = \{(0,y) \in I | -1 \leq y \leq -\frac{1}{2}\}$, $B = \{(\frac{1}{\pi},0) \in E^2\}$ and $S = \{(0,y) \in I | \frac{1}{2} \leq y \leq 1\}$. Since X is connected, a pair of disjoint closed sets A and B is an essential family by Corollary 2.1. Every continuum connecting A and B meets S. Hence, S is a continuum-wise separator. But it is easy to check that S is not a separator of A and B in X.

Assume $\{(A_i,B_i) | i \in \Delta\}$, $|\Delta| > 2$ is an essential family of a compact Hausdorff space X such that a continuum-wise separator S of A_i and B_i does not meet both A_i and B_i. Then is it true that S is a separator? Again, we have a negative example:

Example 2.3. Consider a continuum X which is obtained by extending the Figure 1 of Example 2.2 to z axis in E^3 up to the unit interval [0,1] in the following way:

Let $I^2 = \{(x,y,z) \in E^3 | x = 0, -1 \leq y \leq 1, 0 \leq z \leq 1\}$,

$Y = \{(x,y,z) \in E^3 | y = \sin \frac{1}{x}, 0 < x \leq \frac{1}{\pi}, 0 \leq z \leq 1\}$ and $X = I^2 \cup Y$.

Let $A_1 = \{(0,1,z) \in I^2 | 0 \leq z \leq 1\}$, $B_1 = \{(0,-1,z) \in I^2 | 0 \leq z \leq 1\}$,

$A_2 = \{(0,y,0) \in I^2 | -1 \leq y \leq 1\}$ and $B_2 = \{(0,y,1) \in I^2 | -1 \leq y \leq 1\}$.

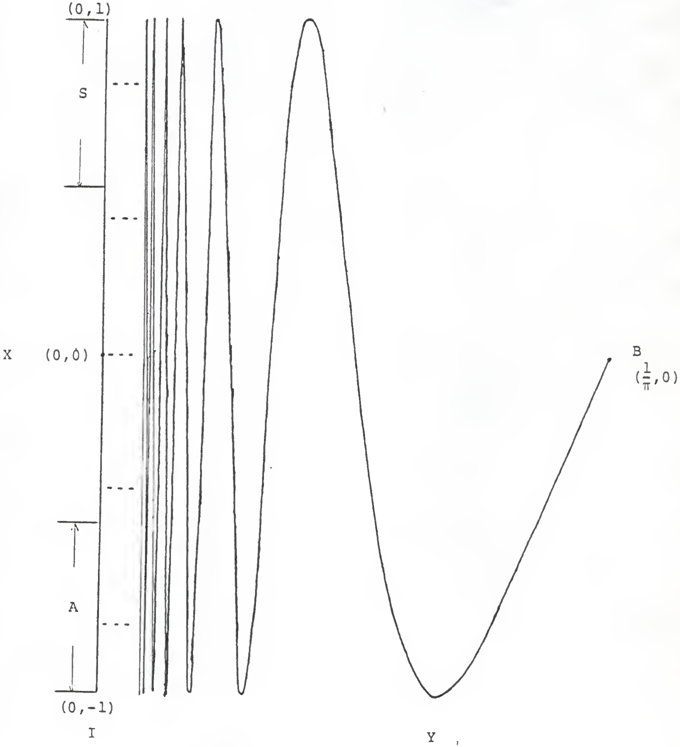


Figure 1.

Then $\{(A_i,B_i) | i = 1,2\}$ is an essential family in X and $S = \{(0,0,z) \in I^2 | 0 \leq z \leq 1\}$ is a continuum-wise separator of A_1 and B_1. However, S is not a separator. We may notice that a point $p = (0, \frac{1}{2}, \frac{1}{2})$ on I^2 is not a continuum-wise separator of $A = \{(0,y,z) \in I^2 | -1 \leq y \leq -\frac{1}{2}, 0 \leq z \leq 1\}$ and $B = \{(\frac{1}{\pi},0,z) \in E^3 | 0 \leq z \leq 1\}$ since $\{(0,y,0) \in I^2 | -1 \leq y \leq 1\} \cup \{(x,y,0) \in Y | y = \sin \frac{1}{x}, 0 < x \leq \frac{1}{\pi}\}$ is a continuum which does not meet a point p, although a point $(0, \frac{1}{2})$ on I in Example 2.2 is a continuum-wise separator of A and B.

Note that the spaces of the above examples are not locally connected. Concerning the positive situation, we have the following lemma.

Lemma 2.2. Let S be a continuum-wise separator of disjoint closed sets A and B in a locally connected compact Hausdorff space X. Assume S misses A and B. Then S is a separator.

Proof. Since S is closed, $X \setminus S$ is again locally connected. Let $C(A) = \{A_i | i \in \Delta\}$ be a set of all continua in $X \setminus S$ which meet A. Set $U = \cup C(A)$. Then U is open in $X \setminus S$ by the local connectedness. Now none of the members of C(A) meets B, because otherwise the member which meets B must meet S by the definition of the continuum-wise separator. However, this contradicts that the member is contained in $X \setminus S$. It follows that $U \cap B = \emptyset$. Let us claim $U \cup S = \overline{U} \cup S$. $U \cup S \subset \overline{U} \cup S$ is trivial. In order to prove the converse inclusion, we shall show $\overline{U}\setminus U \subset S$. Let us take an element $a \in \overline{U}\setminus U$. If $a \notin S$, then $a \in X\setminus S$. However, for any closed connected neighborhood V of a with $V \cap S = \emptyset$ we have $V \cap U \neq \emptyset$ since $a \in \overline{U}$. Hence, there is a continuum $A_i \in C(A)$ with $A_i \cap V \neq \emptyset$. Since X is compact, $A_i \cup V$ is a continuum contained in $X\setminus S$. Hence, $A \cup V \in C(A)$, a fortiori, $a \in U$, a contradiction. Therefore, we have proved $U \cup S = \overline{U} \cup S = \overline{U \cup S}$, which implies $U \cup S$ is closed in X. Since $(U \cup S) \cap B = \emptyset$, $W = X\setminus(U \cup S)$ is an open set containing B. So we have a separation $X\setminus S = U \cup W$ with $A \subset U$ and $B \subset W$, i.e., S is a separator.

This proof works for a locally connected, locally compact Hausdorff space X provided the connected neighborhood V is taken as a compact subset. Hence, we have

Theorem 2.3. Let A and B be disjoint closed subsets in a locally connected, locally compact Hausdorff space X. Assume S is a continuum-wise separator missing A and B. Then S is a separator.

We have the following theorem outside the compact Hausdorff spaces.

Theorem 2.4. Let X be a topological space. Then a pair of non-empty disjoint compact sets A and B is an essential family if and only if there is a quasi-component of X which meets A and B.

Proof. Sufficiency: Assume there is a quasi-component C meeting A and B. If the empty set separates A and B, there exists an open-and-closed set U such that $A \subset U$ and $U \cap B = \emptyset$. Since C is a quasi-component and since $C \cap U \supset C \cap A \neq \emptyset$, we have $U \supset C$. But this contradicts the fact that $U \cap B = \emptyset$ since $U \cap B \supset C \cap B \neq \emptyset$.

Necessity: Assume there is no quasi-component connecting A and B. Let $C(A) = \{A_i \mid i \in \Delta\}$ be the set of all quasi-components of X which meet A. Then none of the quasi-components meets B by our assumption. Pick an arbitrary element $a \in A$. Then there is a quasi-component $A_i \in C(A)$ with $a \in A_i$, which is an intersection of open-and-closed sets $\{A_{ij} \mid j \in \Delta_i\}$. Since $\bigcap_{j \in \Delta_i} (B \cap A_{ij}) = B \cap (\bigcap_{j \in \Delta_i} A_{ij}) = B \cap A_i = \emptyset$, and since each $B \cap A_{ij}$ is a closed subset, by the finite intersection property of a compact set B there is a finite number of indices $j \in \Delta_i$, say $\{1,2,\ldots,n\} \subset \Delta_i$ such that $\bigcap_{k=1}^{n} (B \cap A_{ik}) = \emptyset$. Let $0_i = \bigcap_{k=1}^{n} A_{ik}$. Then 0_i is an open-and-closed subset with $a \in A_i \subset 0_i \subset X \setminus B$. Hence, A is covered by such open-and-closed sets $\{0_i\}$. By the compactness of A, A is covered by a finite number of such sets, say $\{0_j\}_{j=1}^{m}$. Then $0 = \bigcup_{j=1}^{m} 0_j$ is an open-and-closed set with $A \subset 0 \subset X \setminus B$. Hence, $\{A,B\}$ is not an essential family. This proves the necessity.

Since each point in a T_1 space is compact, we have the following corollary.

Corollary 2.3. A T_1-topological space X has dimension ≥ 1 if and only if X has a quasi-component which contains two different points.

Actually, we have proved Ind $X \geq 1$ in Corollary 2.3. According to Nagami ([26], Dimension theory, Theorem 9-9), the covering dimension of X is also ≥ 1 provided X is a normal space. However, we can consider Corollary 2.3 makes possible to define dimension ≥ 1 outside normal spaces.

Theorem 2.5. Let $\{S_i\}$ be a sequence of continuum-wise separators of disjoint closed sets A and B in a compact metric space X. Assume S_i converges to S. Then S is a continuum-wise separator.

Proof. Let C be a continuum which meets A and B. Then $C \cap S_i \neq \emptyset$ for every i. Since X is compact, $\lim C \cap S_i = C \cap S \neq \emptyset$, i.e., S is a continuum-wise separator.

By Lemma 2.1 and Theorem 2.5, we have

Corollary 2.4. Let $\{S_i\}$ be a sequence of separators of disjoint closed sets A and B in a compact metric space X. Assume S_i converges to S. Then S is a continuum-wise separator.

The following simple example shows that the conclusion of Corollary 2.4 cannot be improved to a separator.

Example 2.4. Let $I^2 = \{(x,y) \in E^2 | 0 \leqq x \leqq 1,\ 0 \leqq y \leqq 1\}$, $A = \{(x,y) \in I^2 | x = 0\}$, $B = \{(x,y) \in I^2 | x = 1\}$ and $S_n = \{(x,y) \in I^2 | x = \frac{1}{n}\}$, $n = 2,3,4,\ldots$. Then $\{S_n\}_{n=2}^{\infty}$ is a sequence of separators of A and B which converges to A. But A is not a separator; nevertheless, it is a continuum-wise separator.

Remark 2.2. When we consider the convergence of a sequence of non-empty closed subsets in a compact metric space, we consider the limit defined by Theorem 1.1. However, we can also consider the limit with respect to the topology generated by the Hausdorff metric on the hyperspace 2^X of all non-empty closed subsets of X. Both limits coincide as long as the original space X is compact metric. Then the hyperspace 2^X is also compact metric. We shall prove it in the Appendix.

By virtue of Theorems 2.2 and 2.5, the set S(A,B) of all continuum-wise separators of an essential family $\{(A,B)\}$ of a compact metric space X is a closed subset of the compact hyperspace 2^X; hence, S(A,B) itself is compact metric and has countable dense continuum-wise separators with respect to the Hausdorff metric. The set S'(A,B) of all separators of an essential family $\{(A,B)\}$ is not closed in 2^X. However, it is separable metric as a subspace of S(A,B); hence, S'(A,B) has also countable dense separators with respect to the Hausdorff metric.

Remark 2.3. In Examples 2.2 and 2.3, each continuum-wise separator S is the limit of a sequence of separators of A and B in X. So we have the following question. Does every continuum-wise separator contain a closed subset which is a limit of a sequence of separators in a compact metric space? We give an affirmative answer in several steps.

Lemma 2.3. Let S be a continuum-wise separator of non-empty disjoint closed subsets A and B in a compact Hausdorff space X. Assume S misses both A and B. Then any neighborhood of S contains a separator of A and B.

Proof. Let W be any neighborhood of S in X. Since S misses A and B, by the normality of the compact Hausdorff space X, there exists an open neighborhood U of S such that $\overline{U} \subset W$ and $\overline{U}$ misses A and B. We shall show that the boundary $\partial U = \overline{U} \setminus U$ is a required separator. Since $X \setminus U$ is compact Hausdorff and no continuum meets both A and B in $X \setminus U$, we can apply Theorem 2.2 in $X \setminus U$. That is, there exists a subset $0'$ which is open-and-closed in $X \setminus U$ and $A \subset 0' \subset X \setminus B$. Set $0 = 0' \setminus \partial U = 0' \setminus \overline{U}$. Then 0 is open in X, because if we put $0' = (X \setminus U) \cap Q$ where Q is open in X then $0 = 0' \setminus \partial U = (X \setminus (U \cup \partial U)) \cap Q = (X \setminus \overline{U}) \cap Q$ and both $X \setminus \overline{U}$ and Q are open in X. Since $0'$ is closed in a closed set $X \setminus U$, $0'$ is closed in X; hence, $\overline{0} \subset 0' \subset 0 \cup \partial U$, i.e., we have $\partial 0 = \overline{0} \setminus 0 \subset \partial U$. It follows that $0 \cup \partial U$ is a closed set in X such that $(0 \cup \partial U) \cap B = \emptyset$. Since $A \subset 0 \subset X \setminus B$, we have a separation $X \setminus \partial U = 0 \cup (X \setminus (0 \cup \partial U))$. Therefore, ∂U is a separator of A and B with $\partial U \subset W$.

Theorem 2.6. Let S be a continuum-wise separator of non-empty disjoint closed sets A and B in a compact Hausdorff space X. Then any neighborhood of S contains a separator of A and B.

Proof. If either $A \subset S$ or $B \subset S$, say $A \subset S$, then for any neighborhood W of S, there is an open neighborhood O of A with $A \subset O \subset \overline{O} \subset W$ and $\overline{O}$ misses B. Then $S' = \partial O = \overline{O} \setminus O$ is a separator of A and B. Therefore, we assume $A \not\subset S$ and $B \not\subset S$. Let W be a neighborhood of S. Let U_1 and U_2 be disjoint open neighborhoods of $S \cap A$ and $S \cap B$, respectively, with $\overline{U}_1 \cup \overline{U}_2 \subset W$, $A \not\subset U_1$ and $B \not\subset U_2$. Set $T = U_1 \cup U_2$. Consider a compact Hausdorff space $X \setminus T$.

Set $S_1 = S \setminus T$, $A_1 = A \setminus T$ and $B_1 = B \setminus T$. Then S_1 is a continuum-wise separator of non-empty disjoint closed sets A_1 and B_1. By Lemma 2.3, $W \setminus T$ contains a separator S_2 of A_1 and B_1 in $X \setminus T$. Let $(X \setminus T) \setminus S_2 = V_1 \cup V_2$ be a separation with $A_1 \subset V_1$ and $A_2 \subset V_2$ where V_1 and V_2 are open in $X \setminus T$. By the normality, there are open neighborhoods Q_1 and Q_2 of A_1 and B_1, respectively, with $\overline{Q}_1 \cup \overline{Q}_2$ missing S_2 in X. Set $F_1 = \partial U_1 \setminus Q_1$ and $F_2 = \partial U_2 \setminus Q_2$. Define $V_1' = V_1 \setminus (F_1 \cup F_2)$, $V_2' = V_2 \setminus (F_1 \cup F_2)$ and $S_3 = S_2 \cup F_1 \cup F_2$. Then $O_1 = U_1 \cup V_1'$ and $O_2 = U_2 \cup V_2'$ give a separation of X with $X \setminus S_3 = O_1 \cup O_2$, $A \subset O_1$ and $B \subset O_2$. We leave the reader the task to check that O_1 and O_2 are open in X. Clearly, $S_3 \subset W$. This proves the theorem.

Corollary 2.5. Let S be a continuum-wise separator of nonempty disjoint closed sets A and B in a compact metric space (X,d). Then S contains a closed subset which is the limit of a sequence of separators of A and B in X.

Proof. Each neighborhood $N_{1/n}(S) = \{x \in X \mid d(x,S) < \frac{1}{n}\}$ contains a separator S_n of A and B in X. Since X is compact metric, the sequence $\{S_n\}$ has a convergent subsequence $\{S_{n_j}\}_{j=1}^{\infty}$. This limit is closed and contained in S.

Corollary 2.5 gives another definition of a continuum-wise separator; i.e., S is a continuum-wise separator of disjoint closed sets A and B in a compact metric space if and only if it is a closed set containing a limit of a sequence of separators of A and B. Corollary 2.5 also assures the following characterization of an essential family.

Theorem 2.7. Let $\{(A_i,B_i) \mid i \in \Delta\}$ be a family of pairs of disjoint closed subsets of a compact metric space X. Then $\{(A_i,B_i) \mid i \in \Delta\}$ is an essential family if and only if for any continuum-wise separators S_i of A_i and B_i,

$$\bigcap_{i \in \Delta} S_i \neq 0.$$

Proof. Necessity: Since each continuum-wise separator S_i contains a closed subset S_i' which is a limit of a sequence of separators $\{S_{ij}'\}_{j=1}^{\infty}$ of A_i and B_i, and since $\{(A_i,B_i)\}$ is an essential family, we have for each j

$$\bigcap_{i\in\Delta} S'_{ij} \neq \emptyset.$$

It follows from the compactness of X that

$$\bigcap_{i\in\Delta} S_i \supset \bigcap_{i\in\Delta} S'_i = \lim_{j\to\infty} \bigcap_{i\in\Delta} S'_{ij} \neq \emptyset.$$

Sufficiency: This follows from Lemma 2.1.

Corollary 2.6. Let $\{(A_i, B_i) | i \in \Delta\}$ be an essential family in a compact metric space X. Then $\bigcap_{i\in\Delta} C_i \neq \emptyset$, where C_i is either A_i or B_i.

Proof. Each C_i is a continuum-wise separator of A_i and B_i. Hence, the corollary follows from the previous Theorem 2.7.

The following statement is from Rubin, Schori and Walsh ([31], Proposition 5.5). The proof of Theorem 2.8 follows immediately from Theorem 2.7 and Theorem 1.3. The second assertion follows from Theorem 2.2.

Theorem 2.8. Let $\{(A_k, B_k) | k \in \Delta\}$ be an essential family in a compact metric space X. Let $\Delta_1 \subsetneq \Delta$ be a non-empty proper subset of indices Δ. Assume S_k is a continuum-wise separator of A_k and B_k for each k. Letting $X = \bigcap \{S_k : k \in \Delta\setminus\Delta_1\}$, $\{X \cap A_k, X \cap B_k) | k \in \Delta_1\}$ is essential in X; hence, each $k \in \Delta_1$, X contains a continuum meeting A_k and B_k.

By the same reason using the definition of an essential family and Theorem 1.3, we have the following statement.

Theorem 2.9. Let $\{(A_k,B_k) | k \in \Delta\}$ be an essential family in a metric space X. Let $\Delta_1 \subsetneq \Delta$ be a non-empty proper subset of indices Δ. Assume S_k is a separator of A_k and B_k for each k. Letting $X = \bigcap\{S_k | k \in \Delta\setminus\Delta_1\}$, $\{(X \cap A_k, X \cap B_k) | k \in \Delta_1\}$ is essential in X.

We shall state the refinement of Theorem 2.8 by Rubin, Schori and Walsh ([31], Proposition 5.6). The proof is exactly the same.

Corollary 2.7. Let $\{(A_k,B_k) | k \in \Delta\}$ be an essential family in a compact metric space X. Let $\Delta_1 = \{j_1,j_2,\ldots\}$ be a finite or infinite subset of Δ. Let $\{S_j | j \in \Delta_1\}$ be a collection of closed subsets with S_{j_1} continuum-wise separating A_{j_1} and B_{j_1} and with, for $i > 2$, $S_{j_i} \cap S_{j_{i-1}}$ continuum-wise separating $A_{j_i} \cap S_{j_{i-1}}$ and $B_{j_i} \cap S_{j_{i-1}}$ in $S_{j_{i-1}}$. Letting $X = \bigcap\{S_j | j \in \Delta_1\}$ the family $\{(X \cap A_k, X \cap B_k) | k \in \Delta - \Delta_1\}$ is essential in X.

Remark 2.4. A hereditarily infinite dimensional continuum S of a strongly infinite dimensional compact metric space X with an essential family $\{(A_i,B_i) | i = 1,2,3,\ldots\}$ is constructed as the intersection of continuum-wise separators S_i of A_i and B_i for $i \geqq 2$; i.e., as

$$S = \bigcap_{i > 2} S_i$$

by Henderson, Bing, Rubin, Schori and Walsh. Then Theorem 2.8 and Corollary 2.7 are used to assure that $\dim S > 1$ since S contains a continuum meeting A_1 and B_1. As evidence, the reader may look at Henderson ([8], Lemma 1), Bing ([3], Theorem 5), Rubin ([28], Proposition 2.4), and Schori and Walsh ([33], Theorem 2.7).

Combining the theorems stated above, we can easily derive various conclusions which are used to construct hereditarily strongly infinte dimensional spaces by the authors stated above.

The following is from Rubin, Schori and Walsh ([31], Theorem 4.2).

Theorem 2.10. Let $\{(A_0,B_0),(A_i,B_i) | i \in \Delta\}$ be an essential family of a compact metric space X. Let Y be a subset of X. Then $\{(A_i \cap Y, B_i \cap Y | i \in \Delta\}$ is an essential family in Y if Y meets each continuum from A_0 to B_0. Hence, in this case we have $\dim Y > |\Delta|$.

Proof. Assume Y meets each continuum from A_0 to B_0. Let S_i be a separator of $A_i \cap Y$ and $B_i \cap Y$ in Y for each $i \in \Delta$. Then by the separation extension Theorem 1.3, there is a separator S_i' of A_i and B_i in X with $S_i' \cap Y = S_i$ for each i.

Since $\{(A_0,B_0),(A_i,B_i) | i \in \Delta\}$ is an essential family $\bigcap_{i\in\Delta} S_i'$ contains a continuum C connecting A_0 and B_0 by Theorem 2.8. Then $\bigcap_{i=1}^{n} S_i = \bigcap_{i=1}^{n} (S_i' \cap Y) = Y \cap (\bigcap_{i=1}^{n} S_i') \supset Y \cap C \neq \emptyset$. Therefore, $\{(A_i \cap Y, B_i \cap Y) | i \in \Delta\}$ is an essential family.

Rubin, Schori and Walsh used Theorem 2.10 in the case that $|\Delta|$ is a finite cardinality n to construct a totally disconnected separable metric n-dimensional space Y by choosing one element from each continuum meeting A_0 and B_0 so that the selection of points assures the total disconnectedness ([31], Example 4.3). If we apply this choosing in the case $|\Delta| = \aleph_0$, we shall get a totally disconnected strongly infinite-dimensional separable metric space Y (Rubin [30], Proposition 3.1). This Y is chosen so that Y is a G_δ-set of some compactum M (Roman Pol [27], Lemma). Then from this space Roman Pol constructed an example of an infinite dimensional compactum which is non-countable and weakly infinite dimensional as we saw in the previous Section 1.

The following theorem is from Rubin, Schori and Walsh ([31], Proposition 3.4), Schori and Walsh ([33], Proposition 2.5) and Schori ([32], Theorem 4.4). This was used by them to prove that every non-degenerate continuum in their hereditarily strongly infinite-dimensional compacta has dimension ≥ 2.

Theorem 2.11. Let $\{(A_k, B_k) | k = 1,\ldots,n\}$ be an essential family in a compact metric space X. Assume $\mathcal{S}(k)$ is a countable dense set of separators of A_k and B_k in X, and Y is a closed subset of X. If, for each choice of separators, $S_k \in \mathcal{S}(k)$, $k = 1,\ldots,n$ we have that $\bigcap\{S_k: k=1,\ldots,n\} \cap Y \neq \emptyset$, then $\{(A_k \cap Y, B_k \cap Y) | k = 1,\ldots,n\}$ is essential in Y and, therefore, $\dim Y \geq n$.

Proof. Let S_k' be a separator of $A_k \cap Y$ and $B_k \cap Y$ in Y for each $k = 1,\ldots,n$. Then each S_k' is extended to a separator S_k of A_k and B_k in X with $S_k \cap Y = S_k'$ by Theorem 1.3. Since $\mathcal{S}(k)$ is a dense set, there is a sequence $\{S_{ki}\}_{i=1}^{\infty} \subset \mathcal{S}(k)$ such that $S_{ki} \to S_k$ as $i \to \infty$. By our hypothesis, $\bigcap_k \{S_{ki} | k = 1,\ldots,n\} \cap Y \neq \emptyset$ for each i. By virtue of the compactness of X, we have $\bigcap_{k=1}^{n} S_k' = (\bigcap_{k=1}^{n} S_k) \cap Y = (\lim_i \bigcap_{k=1}^{n} S_{ki}) \cap Y = \lim_i ((\bigcap_{k=1}^{n} S_{ki}) \cap Y) \neq \emptyset$. Hence, $\{(A_k \cap Y, B_k \cap Y) | k = 1,\ldots,n\}$ is essential in Y. This proves the theorem.

In the next section, we shall give an example of a hereditarily infinite dimensional compactum in order to explain how Theorem 2.11 is used to prove that every non-degenerate continuum has dimension > 2.

The following corollary is a modification of Henderson ([8], Lemma 2), and Bing ([3], Theorem 2). It is used to prove that the compact metric space S constructed in Remark

2.4 as the intersection of continuum-wise separators by Henderson and Bing in a compact metric space X has no positive finite dimensional closed subset. If it had, then every positive finite dimensional closed subset would contain a non-degenerate 1-dimensional continuum. However, this corollary is used to prove that every non-degenerate continuum constructed by Henderson and Bing must have a dimension ≥ 2. Since dim S ≥ 1 by Remark 2.4, we can conclude that S is a hereditarily infinite dimensional compactum. Our proof of the corollary is different from Henderson and Bing.

Corollary 2.8. Let $\{(A_i,B_i)\mid i = 1,2\}$ be pairs of disjoint closed sets in a compact metric space X. Let Y be a compact subset of X. If every separator of A_1 and B_1 in X intersects Y in a set containing a continuum from A_2 to B_2, then $\{(A_i\cap Y,B_i\cap Y)\mid i = 1,2\}$ is an essential family in Y; hence, dim Y ≥ 2.

Proof. Let S_1 be a separator of $A_1\cap Y$ and $B_1\cap Y$ in Y. Then S_1 extends to a separator S_1' of A_1 and B_1 in X with $S_1'\cap Y = S_1$ by Theorem 1.3. It follows from our hypothesis that there is a continuum C meeting A_2 and B_2 with $C\subset S_1'\cap Y = S_1$. Let S_2 be a separator of $A_2\cap Y$ and $B_2\cap Y$ in Y. Then

$$S_1\cap S_2\supset C\cap S_2 \neq \emptyset,$$

because, otherwise S_2 gives a separation of a continuum C, which contradicts the connectedness of C. This proves $\{(A_i \cap Y, B_i \cap Y) \mid i = 1,2\}$ is an essential family in Y.

We shall explain the example of Bing [3] in the next section and show how Corollary 2.8 is used to prove that every non-degenerate continuum has a dimension ≥ 2.

We shall finish this section by stating the important correspondence of an essential family and an essential map to a cell I^n. Several theorems concerning the infinte dimensional compacta of Henderson and Bing are proved in the language of an essential map. Using this correspondence these proofs can be written using the language of an essential family as we did and will do in this section. We shall start with the definition.

Definition 2.2. A map f of a metric space X onto an n-cell I^n is said to be inessential if there is a map $g: X \to \partial I^n = S^{n-1}$ such that $f = g$ on $f^{-1}(\partial I^n)$. If there is no such map g, we say that f is essential (or stable).

The following theorem is standard, and the proof is found in usual topology books ([5], Lemma 1.9.7).

Theorem 2.12. (Borsuk's homotopy extension theorem). Let X be a topological space such that X x I is normal (in particular, a metric or a compact Hausdorff space), F a closed set of X, and f and g mappings from F to S^n. If f is homotopic to g and g can be extended over X, then f can also be extended over X.∎

Now we state our correspondence theorem. This theorem is proved in usual dimension theory books in various forms. Our current form is taken from Walsh ([37], Proposition 4.3).

Theorem 2.13. Let X be a metric space and $\{(A_k,B_k) | k = 1, \ldots, n\}$ a family of pairs of disjoint closed subsets of X_n. Let $f_k:X \to I_k = [-1,1]$ be mappings with $A_k = f_k^{-1}(1)$ and $B_k = f_k^{-1}(-1)$. The family $\{(A_k,B_k) | k = 1,\ldots,n\}$ is essential if and only if the mapping $f:X \to I^n = I_1 \times \ldots \times I_n$ defined by $f = (f_1,f_2,\ldots,f_n)$ is essential.

Proof. Necessity: Assume $\{(A_k,B_k) | k=1,\ldots,n\}$ is essential. Set $K = \bigcup_{k=1}^{n} (A_k \cup B_k)$. Then K is a closed subset of X with $K = f^{-1}(\partial I^n)$. Deny the conclusion and assume $f|K$ extends to $g:X \to \partial I^n = S^{n-1}$. Let $\pi_k:I^n \to I_k$ be a projection. Consider mappings $g_k = \pi_k \circ g:X \to I_k$, $k = 1,\ldots,n$. Set $S_k = g_k^{-1}(0)$. Then S_k separates A_k and B_k in X. Since $0 = (0,\ldots,0) \notin S^{n-1}$, we have $0 \notin g(X) = (g_1(X),\ldots,g_n(X))$. This implies $\bigcap_{k=1}^{n} S_k = \bigcap_{k=1}^{n} (g_k^{-1}(0)) = g^{-1}((0,\ldots,0)) = \emptyset$, a contradiction to the essentiality of $\{(A_k,B_k) | k = 1,\ldots,n\}$. Hence, f is essential.

Sufficiency: Assume f is essential. If $\{(A_k,B_k) | k = 1, \ldots,n\}$ is not essential, there are separators S_k of A_k and B_k with $\bigcap_{k=1}^{n} S_k = \emptyset$. Since X is metric, there are mappings

$g_k:X \to I_k$ with $A_k = g_k^{-1}(1)$, $B_k = g_k^{-1}(-1)$ and $S_k = g_k^{-1}(0)$. Set $g = (g_1,\dots,g_n)$. Since $\bigcap_{k=1}^{n} S_k = \emptyset$, $g(X) = (g_1(X),\dots, g_n(X)) \not\ni (0,\dots,0) = 0$. Let r be a radial projection $r:I^n - (0,\dots,0) \to S^{n-1} = \partial I^n$. Then for any $x \in K$, there is a k with $x \in A_k \cup B_k$. For simplicity, assume $x \in A_k$. Then $g_k(x) = 1 = f_k(x)$; i.e., $g(x)$ and $f(x)$ belong to the same surface face of I^n for any $x \in K$. Hence, g and f are homotopic on K. Since $r\circ g = g$ on K, and since $r\circ g:X \to S^{n-1}$ is an extension of $r\circ g|K:K \to S^{n-1}$, it follows from the Borsuk's extension Theorem 2.12 that f can be extended to a mapping $\tilde{f}:X \to S^{n-1}$. This contradicts the essentiality of f. Hence, $\{(A_k,B_k)|k = 1,\dots,n\}$ is essential.

The following corollary is obvious from the above proof.

Corollary 2.9. Let X be a metric space and $\{(A_k,B_k): k = 1,\dots,n\}$ be a family of pairs of disjoint closed subsets of X. Let $f_k:X \to I_k = [-1,1]$ be a mapping with $A_k = f_k^{-1}(1)$ and $B_k = f_k^{-1}(-1)$. Set $K = \bigcup_{k=1}^{n} (A_k \cup B_k)$ and $f = (f_1,f_2,\dots, f_n)$. Then the family $\{(A_k,B_k)|k = 1,\dots,n\}$ is essential if and only if the mapping $f:K \to S^{n-1} = \partial I^n$ cannot be extended to $\tilde{f}:X \to S^{n-1}$.

Corollary 2.9 is used to decide an essential family. The following example is from the class lecture of D.C. Wilson.

Example 2.5. We shall give an essential family of an annulus in E^2. Let X be an annulus surrounded outside by the circle S_1 with vertices $\alpha_1, \beta_1, \gamma_1, \delta_1$ in the counter-clockwise order and inside by the circle S_2 with vertices $\alpha_2, \beta_2, \gamma_2, \delta_2$ in the clockwise order. Put edges $A_1' = \alpha_1\beta_1$, $A_2' = \beta_1\gamma_1$, $B_1' = \gamma_1\delta_1$, $B_2' = \delta_1\alpha_1$, $A_1'' = \alpha_2\beta_2$, $A_2'' = \beta_2\gamma_2$, $B_1'' = \gamma_2\delta_2$, $B_2'' = \delta_2\alpha_2$, and put $A_1 = A_1' \cup A_1''$, $A_2 = A_2' \cup A_2''$, $B_1 = B_1' \cup B_1''$, $B_2 = B_2' \cup B_2''$. (See Figure 2.)

Then $S_1 = A_1' \cup A_2' \cup B_1' \cup B_2'$ and $S_2 = A_1'' \cup A_2'' \cup B_1'' \cup B_2''$. We claim that $\{(A_k, B_k) | k = 1,2\}$ is an essential family of the annulus X. Let $\alpha = (1,-1)$, $\beta = (1,1)$, $\gamma = (-1,1)$ and $\delta = (-1,-1)$ be vertices of $S^1 = \partial I^2$. Let us define $f(\alpha_k) = \alpha$, $f(\beta_k) = \beta$, $f(\gamma_k) = \gamma$, $f(\delta_k) = \delta$; $k = 1,2$ and f maps the edges between vertices homeomorphically to the corresponding edges, for example, the edge $A_1' = \alpha_1\beta_1$ is mapped to the edge $\alpha\beta$ by f. Let $\pi_k : I^2 \to I_k$, $k = 1,2$ be a projection, let $f_k = \pi_k \circ f$. Then we have $A_k = f_k^{-1}(1)$ and $B_k = f_k^{-1}(-1)$. If the map $f : K \to S^1 = \partial I^2$ extended to $\tilde{f} : X \approx S_1 \times I \to S^1$, then $\tilde{f}$ would define a homotopy $\tilde{f} : S_1 \times I \to S^1$ with $\tilde{f}(S_1, 0) = f(S_1)$ and $\tilde{f}(S_1, 1) = f(S_2)$. Since deg $f|S_1 = 1$ and deg $f|S_2 = -1$, this would contradict Hopf's homotopy mapping theorem, which states that the mappings between n-spheres are homotopic if and only if the degrees of the mappings are equal. Therefore, f cannot be extended to

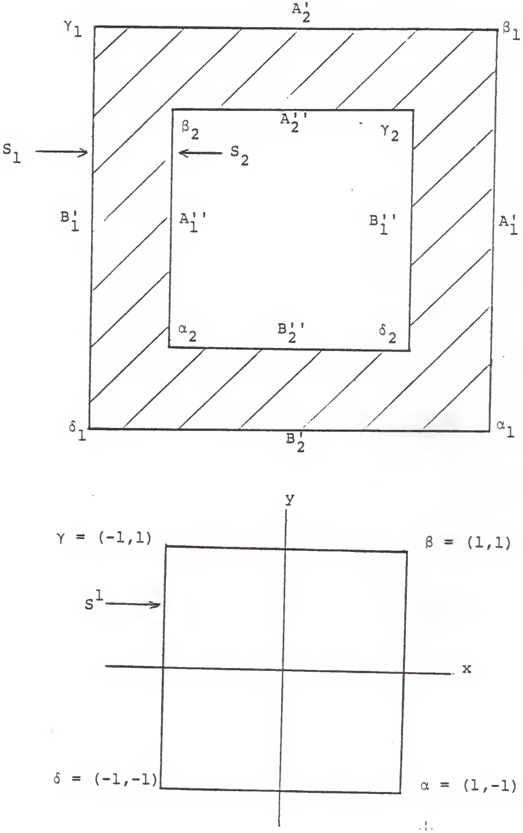

A_2'
γ_1
β_1
A_2''
β_2
γ_2
S_1
S_2
B_1'
A_1''
B_1''
A_1'
α_2
B_2''
δ_2
δ_1
B_2'
α_1
y
$\gamma = (-1,1)$
$\beta = (1,1)$
S^1
x
$\delta = (-1,-1)$
$\alpha = (1,-1)$

Figure 2.

a mapping of the annulus X. It follows from Corollary 2.9 that $\{(A_k, B_k) | k = 1,2\}$ is an essential family.

As an application of Theorem 2.13, we shall give several corollaries which were proved by Bing and Henderson using essential mappings.

Our first corollory is the extension of Henderson ([8], Lemma 1), and Bing ([3], Theorems 3 and 4).

Corollary 2.10. Suppose f is an essential map of a compact metric space X onto I^{n+m}; $n,m \geq 1$, and $\pi\langle 1,2,\ldots,n\rangle$ is the projection map of I^{n+m} onto I^n. Let S_{n+i}; $i = 1,\ldots,m$ be continuum-wise separators between $(\pi_{n+i} \circ f)^{-1}(1)$ and $(\pi_{n+i} \circ f)^{-1}(-1)$, where $\pi_{n+i}: I^{n+m} \to I_{n+i} = [-1,1]$ is a projection. Put $Y = \bigcap_{i=1}^{n} S_{n+i}$. Then $\pi\langle 1,2,\ldots,n\rangle \circ f|Y$ is an essential map of Y onto I^n.

Proof. Let $A_k = (\pi_k \circ f)^{-1}(1)$ and $B_k = (\pi_k \circ f)^{-1}(-1)$. Then by Theorem 2.13, $\{(A_k, B_k) | 1 \leq k \leq n+m\}$ is an essential family. By virtue of Theorem 2.8, $\{(Y \cap A_k, Y \cap B_k) | k = 1,\ldots,n\}$ is essential in Y. Now $Y \cap A_k = (\pi_k \circ f)^{-1}(1) \cap Y = (\pi_k \circ f|Y)^{-1}(1)$, $Y \cap B_k = (\pi_k \circ f|Y)^{-1}(-1)$, and $(\pi_1 \circ f|Y, \pi_2 \circ f|Y, \ldots, \pi_n \circ f|Y) = \pi\langle 1,2,\ldots,n\rangle \circ f|Y$. Hence, again by Theorem 2.13, $\pi\langle 1,2,\ldots,n\rangle \circ f|Y: Y \to I^n$ is an essential map.

Corollary 2.10 is true for the same reason using Theorems 2.9 and 2.13 if we enlarge a space X to a metric space and reduce continuum-wise separators to separators.

Our next corollary is from Henderson ([8], Lemma 5).

Corollary 2.11. Assume X is compact metric. Let $f:X \to I^{n+m} = I^n \times I^m$; $n,m > 1$ be essential. Then for each $p \in I^m$, the map $f|f^{-1}(I^n \times \{p\}): f^{-1}(I^n \times \{p\}) \to I^n \times \{p\}$ is essential.

Proof. This follows from Corollary 2.10 since $(\pi_{n+i} \circ f)^{-1}(p_i)$ is a continuum-wise separator between $(\pi_{n+i} \circ f)^{-1}(1)$ and $(\pi_{n+i} \circ f)^{-1}(-1)$ for $1 < i < m$, where $p = (p_1,\ldots,p_m) \in I^m$, and since $f^{-1}(I^n \times \{p\}) = \bigcap_{i=1}^{m} f^{-1} \circ \pi_{n+i}^{-1} (p_i)$.

We have re-proved, from our point of view, almost all the theorems concerning separators, continuum-wise separators and essential families which were previously proved by Henderson, Bing, Rubin, Schori and Walsh. We hope our proof is useful to clarify the properties and relationships of separators, continuum-wise separators and essential families.

Finally, as an application of Theorem 2.13, we consider the following problem concerning the cartesian product of essential families. Let $\{(A_i,B_i)|i = 1,2,\ldots,m\}$ and $\{(C_i,D_i)|i = 1,\ldots,n\}$ be essential families in metric spaces X and Y respectively. Is it true that $\{(A_i \times Y,B_i \times Y)| i = 1,\ldots,m\} \cup \{(X \times C_j,X \times D_j)|j = 1,\ldots,n\}$ is essential?

We shall give an easy counter-example.

Example 2.6. Let $X = Y = R^\omega$ be a set of points in Hilbert space (ℓ^2,d) all of whose coordinates are rational. A =

$\{0 \in \ell^2\} = \{(0,0,0,\ldots) \in \ell^2\}$ and $B = \{x \in \ell^2 | d(x,0) \geq \alpha\}$ for a fixed positvie number $\alpha > 0$. Then $\{(A,B)\}$ is an essential family in R^ω by the proof of Hurewicz and Wallman ([14], Example II, 11). However, $\{(A \times R^\omega, B \times R^\omega), (R^\omega \times A, R^\omega \times B)\}$ is not essential in $R^\omega \times R^\omega$. Because if it were, then dim $R^\omega \times R^\omega$ would be ≥ 2, but $R^\omega \times R^\omega$ is homeomorphic to R^ω and dim $R^\omega \times R^\omega$ = dim $R^\omega = 1$ ([14], Example III, 5).

Note that in Example 2.6, $X = Y = R^\omega$ is separable metric. Pontrjagin's example of two compact 2-dimensional spaces whose product is 3-dimensional suggests to us that the above problem is false even if both X and Y are compact.

However, we know that dim $(X \times I^n)$ = dim X + dim I^n = dim X + n for any paracompact Hausdorff space X ([26], Theorem 42-3). So we try to investigate the above problem in the case of X is metric and Y is a cube I^n. First let us give a definition.

Definition 2.3. A mapping $f:X \to S^n$ is essential with respect to $f^{-1}(c)$ if f cannot be homotopic to a constant map $c:X \to c \in S^n$ with respect to $f^{-1}(c)$, where S^n is a n-sphere.

Let X be an m-dimensional metric space with an essential family $\{(A_i,B_i)\}_{i=1}^m$. Let $f_i:X \to I_i = [-1,1]$, $i = 1,\ldots,m$ be mappings with $A_i = f_i^{-1}(1)$ and $B_i = f_i^{-1}(-1)$. Then $f = (f_1,\ldots,f_m):X \to I^m = I_1 \times \ldots \times I_m$ is an essential map by Theorem 2.13. Set $K = \bigcup_{i=1}^{m} (A_i \cup B_i)$. Let $p:I^m \to I^m/\partial I^m$

$= S^m$ be a quotient map with $p(\partial I^m) = c$. Assume the map $\overline{f}: X/K \to S^m = I^m/\partial I^m$ induced by f is essential with respect to $\overline{f}^{-1}(c) = K$ (refer to [12], Alexandroff's Theorem). Then,

Theorem 2.14. Under the situation mentioned above $\{(A_i \times I, B_i \times I)\}_{i=1}^m$ $\{(X \times \{-1\}, X \times \{1\})\}$ is an essential family in $X \times I$, where $I = [-1,1]$.

Proof. Let $f \times id: X \times I \to I^m \times I$ be a product map, where $id: I \to I$ is identity. Set $K' = K \times I \cup X \times \{1\} \cup X \times \{-1\}$. Then $K' = (f \times id)^{-1}(\partial(I^m \times I))$. In order to prove the essentiality of the family in Theorem 2.14, it suffices to prove the essentiality of the map $f \times id$ by Theorem 2.13. Take a point $c \in I^m \times \{1\}$. Set $S^m = \partial(I^m \times I)$. Notice that we identify c and S^m defined here with those defined in the statement above Theorem 2.14 by some homeomorphism between m-spheres. We can define a homotopy $h: S^m \times I \to S^m$ such that

$$h_{-1}(x) = x \qquad \text{for } x \in S^m,$$

$$h_1(x) = c \qquad \text{for } x \in \bigcup_{i=1}^{m} (A_i \times I \cup B_i \times I) \cup (X \times \{1\}),$$

and $h_1|((\mathrm{Int}(I^m)) \times \{-1\})$ is a homeomorphism.

Then $g = h_1 \circ (f \times id)|K'$ is homotopic to $f \times id|K'$. For the sake of simplicity we identify $h_1 \circ f \times (-1)$ with f by the homeomorphism $h_1|((\mathrm{Int}(I^m)) \times \{-1\})$. They are different in the boundary, but they induce the same $\overline{f}:X/K \to S^m$ with $\overline{f}^{-1}(c) = K$. Now we shall show that $g:K' \to S^m$ cannot be extended to $G:X \times I \to I^m$. Contrary, assume it can. Then $G:X \times I \to S^m$ induces a map $F:(X/K) \times I \to S^m$ such that $F(\{K\} \times I) = c$, $F((X/K) \times \{1\}) = c$ and $F(z,-1) = \overline{f}(z)$. Then F gives a homotopy between $\overline{f}$ and a constant map c with respect to $\overline{f}^{-1}(c) = K$. This contradicts the essentiality of $\overline{f}$. Hence, g cannot be extended to $G:X \times I \to S^m$. This proves the essentiality of $f \times \mathrm{id}$.

By the induction n, we have the following corollary.

Corollary 2.12. Under the situation of Theorem 2.14, $\{(A_i \times I^n, B_i \times I^n)\}_{i=1}^m \cup \{(X \times \{1\} \times I_2 \times \ldots \times I_n, X \times \{-1\} \times I_2 \times \ldots \times I_n), \ldots, (X \times I_1 \times \ldots \times I_{n-1} \times \{1\}, X \times I_1 \times \ldots \times I_{n-1} \times \{-1\})\}$ is essential in $X \times I^n$. Hence, $\dim X \times I^n = \dim X + \dim I^n = \dim X + n$.

Since every n-dimensional polyhedron contains a homeomorphic image of I^n,

Corollary 2.13. Under the situation of Theorem 2.14, for every polyhedron Y, $\dim X \times Y = \dim X + \dim Y$.

Now we have done all our works about the properties of separators, continuum-wise separators and essential families. We are glad if the reader finds the field which he can explore more about these properties.

3. Hereditarily Infinite Dimensional Spaces

This section is aimed to explain how Theorem 2.11 and Corollary 2.8 are used to prove that every non-degenerate continuum has dimension ≥ 2 in the spaces constructed by Henderson, Bing, Rubin, Schori and Walsh. We do not use exactly the same construction. Our constructions are modifications of Bing [3], Rubin, Schori and Walsh [31] and Schori and Walsh [33]. We think the constructions of Schori and Walsh [33] and Schori [32] are much simpler than ours. But we believe our construction proves our aim well. We use the preparation for our examples.

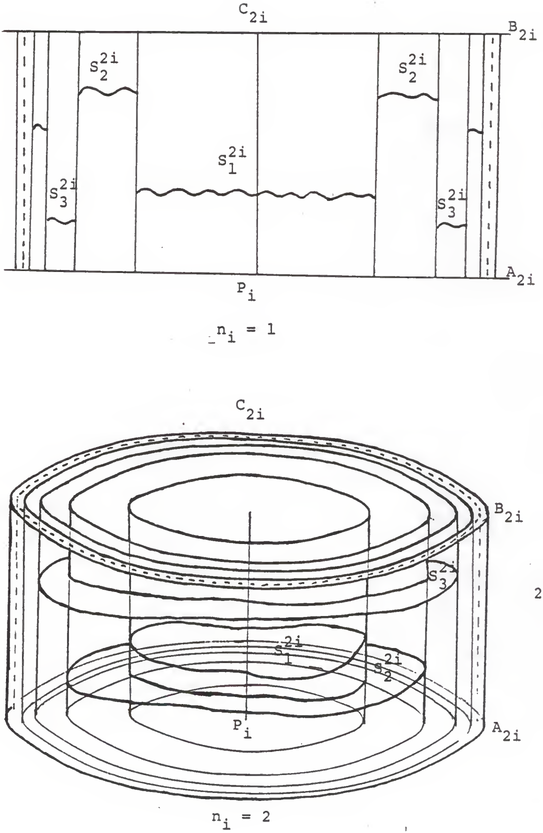
Preparation. We shall construct a hereditarily infinite dimensional continuum S in the Hilbert-Cube $I^\infty = \prod_{i=1}^{\infty} I_i$ as Remark 2.4, where $I_i = [-\frac{1}{2^i}, \frac{1}{2^i}]$ and the Hilbert cube has a metric $d(x,y) = (\Sigma(x_i - y_i)^2)^{\frac{1}{2}}$ between points $x = (x_i)$ and $y = (y_i)$. Furthermore, we assume $I^n = \prod_{i=1}^{n} I_i$ is embedded in the Euclidean n-space E^n in a usual way. Let $\{U_i\}_{i=1}^{\infty}$ be a countable basis of I^∞ with the form

$$U_i = B^{n_i} \times I_{n_i+1} \times I_{n_i+2} \times \cdots \qquad i \geq n_i.$$

where B^{n_i} is the intersection of an open round n_i-ball with a center p_i in E^{n_i} and $I^{n_i} = I_1 \times I_2 \times \ldots \times I_{n_i}$. We assume each U_i, hence each B^{n_i}, is non-empty. Let $d(p_i, E^{n_i} \setminus B^{n_i}) = \varepsilon_i$ in (E^{n_i}, d). Then define $B^{n_i}(j) = \{x \in B^{n_i} | \varepsilon_i - \frac{1}{2^{j-1}} \varepsilon_i < d(p_i, x) < \varepsilon_i - \frac{1}{2^j} \varepsilon_i\}$; $j = 1, 2, \ldots$. Note that by $B^{n_i} \neq \emptyset$ we have $B^{n_i}(j) \neq \emptyset$ for some j on, and each neighborhood of $\partial B^{n_i} = \overline{B^{n_i}} \setminus B^{n_i}$ contains all but finitely many of the $B^{n_i}(j)$, and $\{B^{n_i}(j)\}_{j=1}^{\infty}$ is a sequence of countable disjoint open sets in B^{n_i}. Set $U_{i,j} = B^{n_i}(j) \times I_{n_i+1} \times I_{n_i+2} \times \cdots$. Then $\{U_{i,j} | j = 1, 2, \ldots\}$ is a sequence of countable disjoint open sets in U_i. Let $A_k = \pi_k^{-1}(\frac{1}{2^k})$ and $B_k = \pi_k^{-1}(-\frac{1}{2^k})$ be pairs of opposite faces of the Hilbert cube I^∞, where $\pi_k : I^\infty \to I_k$ is a projection. Then $\{(A_k, B_k) | k = 1, 2, \ldots\}$ is an essential family ([14], p. 40). Let $\mathcal{S}(k) = \{S_j^k | j = 1, 2 \ldots\}$ be a countable dense family of separators of A_k and B_k in I^∞. Our first two examples explain how we apply Theorem 2.11.

Example 2.7. For all even indices $2i$; $i = 1, 2, \ldots$, define a continuum-wise separator C_{2i} of A_{2i} and B_{2i} in I^∞ as follows: $C_{2i} = (I^\infty - \bigcup_{j=1}^{\infty} U_{i,j}) \cup \bigcup_{j=1}^{\infty} \{p | p \in U_{i,j} \cap S_j^{2i}\}$, where $S_j^{2i} \in \mathcal{S}(2i)$. (See Figure 3.)

Let $\{U_{i,j,k}\}_{k=1}^{\infty}$ be countable disjoint open sets in $U_{i,j}$ such that each $U_{i,j,k}$ is non-empty whenever $U_{i,j}$ is. Define



$n_i = 1$

$n_i = 2$

Figure 3.

the continuum-wise separator C_{2i+1} of $C_{2i} \cap A_{2i+1}$ and $C_{2i} \cap B_{2i+1}$ in C_{2i} as follows: $C_{2i+1} = C_{2i} \cap \{ \bigcup_{k=1}^{\infty} (U_{i,j,k} \cap S_k^{2i+1}) \}$ i.e., C_{2i+1} is obtained from C_{2i} by replacing each open set $U_{i,j,k} \cap S_j^{2i}$ by the intersection with the separator S_k^{2i+1} of A_{2i+1} and B_{2i+1} (see [31], Figure 1 for the image of C_{2i+1}). Let $X = \bigcap_{i>2} C_i$. Then X is a compactum with dim $X > 1$ by Remark 2.4. Assume $K \subset X$ is a non-degenerate continuum in X. Then there exists an open set U_i from the basis of I^{∞} such that $K \cap U_i \neq \emptyset$ and $K \cap \partial U_i \neq \emptyset$. By our construction of X, $S_{j_1}^{2i} \cap S_{j_2}^{2i+1} \cap K \neq \emptyset$ for any selections of $S_{j_1}^{2i}$ and $S_{j_2}^{2i+1}$ from $\mathcal{S}(2i)$ and $\mathcal{S}(2i+1)$ respectively. In fact, $K \cap \partial U_i \cap S_{j_1}^{2i} \cap S_{j_1}^{2i+1} \neq \emptyset$. Since the denseness of $\mathcal{S}(2i)$, $\mathcal{S}(2i+1)$ and $\{U_{i,j} \mid j = 1,2,\ldots\}$ around ∂U_i we can find a sequence of intersections of separators each from $\mathcal{S}(2i)$ and $\mathcal{S}(2i+1)$ with U_{i,j_t,k_t} and K, which converges a subset of $K \cap \partial U_i \cap S_{j_1}^{2i} \cap S_{j_2}^{2i+1}$ as $t \to \infty$. Hence, by Theorem 2.11, dim $K > 2$.

Therefore, X is a hereditarily infinite dimensional compactum by the argument before Corollary 2.8.

Example 2.8. We can modify the construction of Example 2.7 in a shortened way if we take $\{T_j^{2i} | j = 1,2,\ldots\}$ as an enumeration of $\{S_{j_1}^{2i} \cap S_{j_2}^{2i+1} | j_1, j_2 > 1\}$ and if we define

$$C_{2i} = (I^\infty - \bigcup_{j=1}^{\infty} U_{i,j}) \cup \bigcup_{j=1}^{\infty} \{p | p \in U_{i,j} \cap T_j^{2i}\}$$

(See Figure 4) and $X = \bigcap_{i=1}^{\infty} C_{2i}$. Then the argument of Example 2.7 shows that X is a hereditarily infinite dimensional compactum.

Now we give an example which explains how Corollary 2.8 is used to prove that every non-degenerate continuum has a dimension > 2. Our Example 2.9 is a modification of Bing [3].

Example 2.9. We use the same notation as Preparation. Set $D^{n_i}(j) = \{x \in U_{i,j} | \pi_{2i}(x) < \frac{1}{2^i}, d(p_i, \pi\langle 1,2,\ldots,n_i\rangle(x)) < \varepsilon_i - \frac{1}{2^{i-1}} \varepsilon_i + \frac{1}{2^{i+1}} \varepsilon_i\} \cup \{x \in U_{i,j} | \pi_{2i}(x) > -\frac{1}{2^i}, d(p_i, \pi\langle 1,2,\ldots,n_i\rangle (x)) > \varepsilon_i - \frac{1}{2^i} \varepsilon_i - \frac{1}{2^{i+1}} \varepsilon_i\}$. Define $U'_{i,j} = U_{i,j} \setminus D^{n_i}(j)$. For all even indices 2i, i = 1,2,..., define $X_{2i} = (I^\infty - \bigcup_{j=1}^{\infty} U_{i,j}) \cup \bigcup_{j=1}^{\infty} \{p | p \in U'_{i,j} \cap S_j^{2i+1}\}$. Note that $X'_{2i} = (I^\infty - \bigcup_{j=1}^{\infty} U_{i,j}) \cup \bigcup_{j=1}^{\infty} \{p | p \in U'_{i,j}\}$ is a continuum-wise separator of A_{2i} and B_{2i} in I^∞ (See Figure 5A) and X_{2i} is a continuum-wise separator of $A_{2i+1} \cap X'_{2i}$ and $B_{2i+1} \cap X'_{2i}$

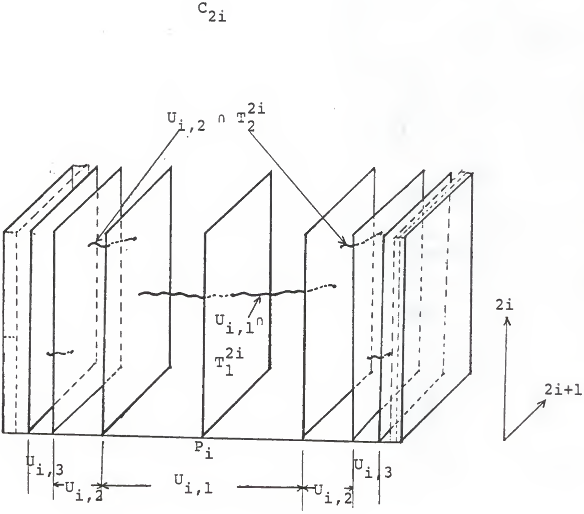


Figure 4.

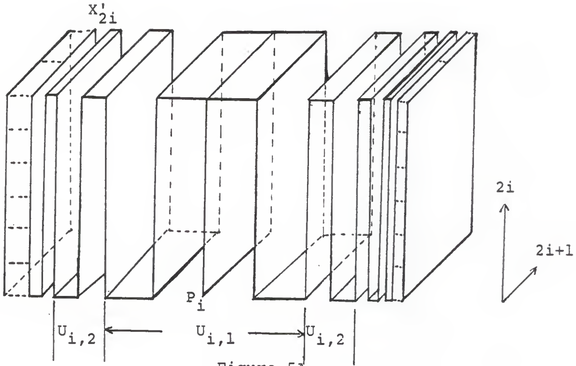


Figure 5A.

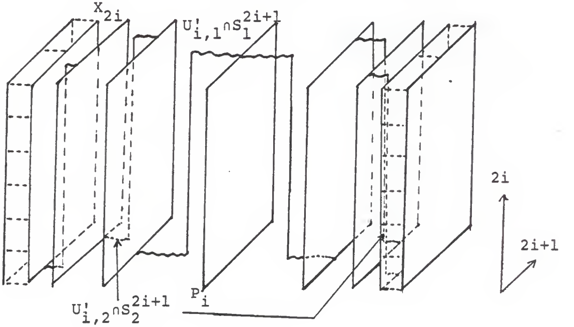


Figure 5B.

in X'_{2i} (See Figure 5B). Also note that $\{(A_k \cap U'_{i,j}, B_k \cap U'_{i,j}) | k = 2i, 2i+1\}$ is an essential family in $U'_{i,j}$. Let $X = \bigcap_{i=1}^{\infty} X_{2i}$. Now we shall prove that every non-degenerate continuum in X has a dimension > 2.

Let $K \subset X$ be a non-degenerate continuum. Then there exists an open set U_i from the basis of I^{∞} such that

$$K \cap U_i \neq \emptyset, \; K \cap \partial U_i \neq \emptyset \; \ldots \; (1).$$

Put $Y = K \cap \partial U_i$. We claim that $\dim Y > 2$ by showing that $\{(A_{2i} \cap Y, B_{2i} \cap Y) | k=2i, 2i+1\}$ is an essential family due to the proof that the hypothesis of Corollary 2.8 is satisfied. Let S be a separator of $A_{2i+1} \cap \partial U_i$ and $B_{2i+1} \cap \partial U_i$ in ∂U_i. Then we want to show that $S \cap K$ contains a continuum joining A_{2i} and B_{2i}. By Theorem 1.3, S extended to a separator S' of A_{2i+1} and B_{2i+1} in I^{∞}. Let $S^{2i+1}_{n_1}$, $S^{2i+1}_{n_2}, \ldots$ be a subsequence of $\{S^{2i+1}_j | j = 1, 2, \ldots\} = \mathcal{S}(2i+1)$ converging to S'. Then $(A_{2i} \cap U'_{i,n_j} \cap S^{2i+1}_{n_j}, B_{2i} \cap U'_{i,n_j} \cap S^{2i+1}_{n_j})$ is an essential family in $U'_{i,n_j} \cap S^{2i+1}_{n_j}$ by Theorem 2.8. Moreover, for sufficiently large j, say $j > j_0$, $A_{2i} \cap U'_{i,n_j} \cap S^{2i+1}_{n_j} \cap K \neq \emptyset$ and $B_{2i} \cap U'_{i,n_j} \cap S^{2i+1}_{n_j} \cap K \neq \emptyset$ by virtue of (1) and $K \subset X$. Hence, for $j > j_0$ there is a continuum $K(n_j)$ in $U'_{i,n_j} \cap$

$S_{n_j}^{2i+1} \cap K$ joining the opposite faces A_{2i} and B_{2i}. Since $\{U_{i,j}\}_{j=1}^{\infty}$ is contained in any neighborhood of ∂U_i except finite numbers we can conclude that some subsequence of $\{K(n_j)\}_{j=j_0}^{\infty}$ converges to a continuum in $S \subset \partial U_i$ by the compactness of a hyperspace 2^X. This continuum lies in $S \cap \partial U_i \cap K$ and joins A_{2i} and B_{2i}. Hence, the hypothesis of Corollary 2.8 is satisfied.

Our Example 2.9 explains the hereditarily infinite dimensional compactum in Bing [3]. We finish this section with the hope that the reader now understands clearly hereditarily infinite dimensional compacta.

CHAPTER III
SUMMARY AND FUTURE STUDY

In Chapter I, we derive a theorem concerning mappings and dimensions. In Chapter II, we derive theorems concerning hereditarily infinite dimensional spaces. These results are extremely different. However, those results are derived from common fundamental theorems, namely, Theorems 1.1 and 1.3. Essential families play an especially important role. This forms the unifying principle in this work.

The main theorems proved in this dissertation are Theorems 1.4, 2.4, 2.7 and 2.13. As we have seen, they are strongly related to the various properties of essential families. Our important tools in analyzing essential families are Theorem 1.1 concerning a limit of a sequence of subsets and Theorem 1.3 concerning an extension of a separator. Since Theorem 1.1 is false in a general metric space, a generalization would seem to need a concept of a net convergence of subsets and some kind of cardinality argument to apply the results obtained here to a space outside separable metric spaces. Theorem 2.2 which is valid for compact metric spaces is obtained from Theorem 2.4 as a special case by virtue of Theorem 2.1. As far as an essential family is concerned, comparing Theorems 2.2 and 2.4, we want to know whether quasi-components play the same role outside the compact metric spaces as continua play in compact metric spaces. Being a continuum is intrinsic; i.e., it does not depend on the space containing it. But, the property of being a quasi-component is heavily dependent upon the space which contains it. Hence, we need some devices to extend Theorems 2.5, 2.6, 2.7 and 2.10 outside the class of compact metric spaces. To this end, the reader needs to define a quasi-separator corresponding to a continuum-wise separator in the case of a compact metric space. We cannot give a workable definition here. But, as an introduction to such a treatment, we shall define a quasi-connected set. A non-empty closed subset K of a topological space X is said to be X-quasi-connected if for every open-and-closed set $U \subset X$ with $K \cap U \neq \emptyset$ we have $K \subset U$. Hence a quasi-connected set is a subset of a quasi-component. We shall give some properties without proof.

(1) Let a closed set $K \subset X$ be X-quasi-connected and U be an open-and-closed set in X with $K \subset U$, then K is U-quasi-connected.

(2) Let A and B be different quasi-components of X. Then there are open-and-closed subsets U_A and U_B such that $A \subset U_A$, $B \subset U_B$ and $U_A \cap U_B = \emptyset$.

(3) Let $Y \subset X$. Assume $K \subset Y$ is a Y-quasi-connected set. Then K is a X-quasi-connected set.

(4) If K is K-quasi-connected, then K is connected.

Perhaps the reader can clarify the possibility of such a generalization.

Now as a future study, we would like to answer the following questions.

(5) Does Theorem 1.4 have an application like that of a Brower fixed-point theorem?

(6) Is the condition of Theorem 2.10 necessary and sufficient?

(7) Are the conditions of Theorem 2.14 satisfied for an m-dimensional metric space X?

Finally we hope that a talented reader will build on this study to accomplish the construction of mathematics similar to that discussed here.

APPENDIX

In our proof, the limit of a sequence of subsets A_i in a metric space X plays an important role. We consider the limit defined by Theorem 1.1. However, if X is a compact metric space, we can consider the limit with respect to the topology generated by the Hausdorff metric on the hyperspace 2^X of all non-empty closed subsets of X. In this appendix, we shall give the rigorous proof that both limits coincide whenever the original space X is compact metric. We start with the definitions.

Let $(2^X,\rho)$ be the hyperspace of a compact metric space (X,d) with the Hausdorff metric ρ defined by $\rho(A,B) = \inf\{\varepsilon > 0 \mid A \subset N_\varepsilon(B) \text{ and } B \subset N_\varepsilon(A)\}$ for any $A,B \in 2^X$, where $N_\varepsilon(A) = \{x \in X \mid d(A,x) < \varepsilon\}$. Assume by $\lim\limits_{i\to\infty} A_i$, $\{A_i\} \subset 2^X$ the limit with respect to the topology generated by this Hausdorff metric. Now we define lim inf A_i and lim sup A_i as follows.

$$\lim_i \inf A_i = \{x \in X \mid \text{for any neighborhood } U_x \text{ of } x \text{ there exists an integer } i_0 \text{ such that } i > i_0 \text{ implies } U_x \cap A_i \neq \emptyset\},$$

$$\lim_i \sup A_i = \{x \in X \mid \text{for any neighborhood } U_x \text{ of } x,\ U_x \cap A_i \neq \emptyset \text{ for infinitely many } i\}.$$

Then for any sequence of subsets $\{A_i\}_{i=1}^{\infty}$ we have the following properties:

(i) $\lim\inf A_i \subset \lim\sup A_i$,

(ii) If $\{A_{i_j}\}_{j=1}^{\infty}$ is cofinal in $\{A_i\}_{i=1}^{\infty}$ then $\lim_i \inf A_i \subset \lim_j \inf A_{i_j}$ and $\lim_i \sup A_i \supset \lim_j \sup A_{i_j}$,

(iii) $\lim\inf A_i$ and $\lim\sup A_i$ are closed subsets of X.

Now we shall give a proof of the following theorem.

Theorem. Let $\{A_k\}$ be a sequence of closed sets in a compact metric space X. Then $\lim\limits_{k\to\infty} A_k = A$ if and only if $\lim_k \sup A_k = \lim_k \inf A_k = A$.

Proof. Assume $\lim\limits_{k\to\infty} A_k = A$. Then $A \in 2^X$, hence A is closed and for any $\varepsilon > 0$ there is an $N > 0$ such that $n > N$ implies $N_\varepsilon(A_n) \supset A$ and $A_n \subset N_\varepsilon(A)$. Take an arbitrary point $x \in A$. Since $N_\varepsilon(A_n) \supset A$, there is an $y_n \in A_n$ with $d(y_n, x) < \varepsilon$. Hence, $N_\varepsilon(x) \cap A_n \neq \emptyset$ for every integer $n > N$. This is true for an arbitrary $x \in A$. Therefore,

$$A \subset \lim\inf A_k. \tag{1}$$

Let $z \notin A$, then $d(A,z) = \varepsilon' > 0$. Take $n_{\varepsilon'}$ such that $n > n_{\varepsilon'}$ implies

$$A_n \subset N_{\varepsilon'/2}(A).$$

Then $N_{\varepsilon'/2}(z) \cap A_n = \emptyset$ for every integer $n > n_{\varepsilon'}$. For, otherwise there is an $x_n \in N_{\varepsilon'/2}(z) \cap A_n$ for some $n > n_{\varepsilon'}$. Since $A_n \subset N_{\varepsilon'/2}(A)$, there is a point $x \in A$ with $d(x,x_n) < \varepsilon'/2$. Then $d(z,x) \leq d(z,x_n) + d(x_n,x) < \frac{\varepsilon'}{2} + \frac{\varepsilon'}{2} = \varepsilon'$, a contradiction to $d(z,A) = \varepsilon'$. Therefore, $z \notin \lim_k \sup A_k$. This is true for any point $z \notin A$. Hence, $A^c \subset (\lim\sup A_k)^c$, that is,

$$A \supset \lim\sup A_k \qquad (2)$$

By (1) and (2)

$$\lim\sup A_k \subset A \subset \lim\inf A_k \qquad (3)$$

By (i) and (3), we have $\lim\inf A_k = A = \lim\sup A_k$.

Sufficiency: Assume $\lim\inf A_k = \lim\sup A_k = A$. Then A is closed by (iii). Given $\varepsilon > 0$. For any point $x \in A$, consider an open neighborhood $N_{\varepsilon/2}(x)$. Since A is compact as a closed subset of a compact space X, A is covered by a finite number of $N_{\varepsilon/2}(x_i)$; $x_i \in A$, $i = 1,\dots,t$, i.e., $A \subset N_{\varepsilon/2}(x_1) \cup N_{\varepsilon/2}(x_2) \cup \dots \cup N_{\varepsilon/2}(x_t)$. Since $x_i \in A$ and $\lim\inf A_k = A$, there is an n_i such that $k > n_i$ implies $A_k \cap N_{\varepsilon/2}(x_i) \neq \emptyset$. Let $N_1 = \max\{n_1,n_2,\dots,n_t\}$. Then for any integer $k > N_1$, $A_k \cap N_{\varepsilon/2}(x_i) \neq \emptyset$ for every $i = 1,2,\dots,t$. Take an arbitrary point $x \in A$, then there is an i with $x \in N_{\varepsilon/2}(x_i)$. Take a point $x_k \in A_k \cap N_{\varepsilon/2}(x_i)$. Then $d(x_k,x) \leq d(x_k,x_i) + d(x_i,x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Since any $x \in A$ is in a distance less than ε from A_k

$$A \subset N_\varepsilon(A_k) \text{ for any integer } k > N_1. \tag{4}$$

Consider $B = X - N_\varepsilon(A)$. Then B is compact as a closed subset of a compact space X. Since $\limsup A_k = A$, for any point $x \in B(\subset A^c)$ there is an open neighborhood U_x of x such that $U_x \cap A_k \neq \emptyset$ for at most finite number of k. Since B is compact, B is covered by a finite number of such neighborhoods, say $U_{x_1}, \dots, U_{x_m}$ $(x_i \in B)$. Let $N_2 = \max \{k \mid A_k \cap U_{x_i} \neq \emptyset$ for some $i \in \{1,2,\dots,m\}\}$. Then for any $k > N_2$ we have $U_{x_i} \cap A_k = \emptyset$ for every $i \in \{1,\dots,m\}$. Since $B \subset U_{x_1} \cup \dots \cup U_{x_m}$, $B \cap A_k = \emptyset$ for every $k > N_2$; i.e.,

$$A_k \subset B^c = N_\varepsilon(A) \text{ for every } k > N_2. \tag{5}$$

Take $N_3 = \max \{N_1, N_2\}$. Then by virtue of (4) and (5), for any $k > N_3$ we have $A_k \subset N_\varepsilon(A)$ and $A \subset N_\varepsilon(A_k)$, hence, $\lim_{k\to\infty} A_k = A$.

Remark. In order to prove necessity, we do not use the compactness. Hence, the convergence with respect to the Hausdorff metric is much stronger than the convergence by means of Theorem 1.1.

By Theorem 1.1 and this theorem we have the following corollary.

Corollary. Let (X,d) be a compact metric space. Then the hyperspace $(2^X,\rho)$ consisting of all nonvacuous closed sets of X is a compact metric space.

Proof. Since 2^X is a metric space under the Hausdorff metric ρ, it is sufficient to prove that every sequence of elements of 2^X has a convergent subsequence. Let $\{A_i\}_{i=1}^{\infty}$ be a sequence of nonvacuous closed subsets of X. Then since every compact metric space is separable, by Theorem 1.1, there is a subsequence $\{A_{i_j}\}_{j=1}^{\infty}$ of $\{A_i\}$ with $\lim_j \inf A_{i_j} = \lim_j \sup A_{i_j} = A$. Since X is compact, $A \neq \emptyset$. For, pick $x_j \in A_{i_j}$ for each j. This is possible since $A_{i_j} \neq \emptyset$. Now $\{x_j\}$ is a sequence of a compact metric space X. Hence, there is a convergent subsequence $\{x_{j_k}\}_{k=1}^{\infty}$ of $\{x_j\}$. Let $x_{j_k} \to x$ as $k \to \infty$. Then $x \in A$. Hence, $A \neq \emptyset$. By (iii), A is closed. Therefore, $A \in 2^X$. By the theorem above, this limit A is the same limit with respect to the Hausdorff metric ρ. Therefore, $\lim_j A_{i_j} = A$ with resepct to ρ. It proves that the hyperspace 2^X is compact.

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BIOGRAPHICAL SKETCH

Eiji Kurihara was born in Hinode-machi, Tokyo, Japan, on October 25, 1948. He completed his secondary education at Itsukaichi High School in Tokyo, Japan, and entered Tokyo Gakugei University in April, 1967, where he obtained a Bachelor of Mathematical Education degree. After teaching mathematics for one year at Mizuho Junior High School, he entered the graduate program of Tokyo Gakugei University. He had studied differential geometry under Dr. Wakakuwa for three years. In the same time, he had the opportunity to listen to a lecture from the famous topologist Kiichi Morita. While he was studying, he taught mathematics to blind students at Hachioji High School for Blind People. In 1974, he finished the master's program in mathematics education. After that he went to Malawi, Central Africa, as a Japan Overseas Cooperation Volunteer and taught mathematics to students of Dowa Day Secondary School in Mponela, Malawi, for two years. He lived alone in a government house with three bedrooms but with no electricity. There was no information from the outside world. He spent nights under the dim Aladdin lamp and watched the stars in the sky. However, this life was the most wonderful time of his youth. Returning to Japan, he taught mathematics to students of various high schools for two years. He also had a chance to help children with learning disabilities at Hachioji High School for Disadvantaged Students. He thanked God for giving him the ability to pursue mathematics. He came to the University of Florida to further such pursuit in 1979. In 1984, he got a chance to publish his first paper in English in the journal General Topology and its Application.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

James Keesling

Dr. James Keesling, Chairman
Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Dr. David Wilson
Associate Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Dr. Irena Lasiecka
Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Dr. James Brooks
Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Dr. George Lebo
Associate Professor of Astronomy

This dissertation was submitted to the Graduate Faculty of the Department of Mathematics in the College of Liberal Arts and Sciences and to the Graduate School, and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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Dean for Graduate Studies and
Research